(Fundamental) Physics of Elementary Particles

Gauss' Laws, Dirac Quantization & Consequences
Dualities and Dual Worldviews
A "Theory" of More Than Everything

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Fundamental Physics of Elementary Particles

Program

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- Dirac's Dual Quantization
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1777–1855
1835–1867
58 12

½

1902–1984
1931
29

Gauss's Laws

E&M Sources and Sinks in 4D

- Recall the Maxwell equations

(Gauss)

Static:

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{4\pi\epsilon_0}4\pi\,\rho_e,$$

$$\vec{\nabla}\cdot(c\vec{B}) = \frac{\mu_0}{4\pi}4\pi\,\rho_m,$$

Dynamic:

$$\vec{\nabla}\times(c\vec{B}) - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{1}{4\pi\epsilon_0}\frac{4\pi}{c}\vec{j}_e, \quad \text{(Ampère)}$$

$$-\vec{\nabla}\times\vec{E} - \frac{1}{c}\frac{\partial(c\vec{B})}{\partial t} = \frac{\mu_0}{4\pi}\frac{4\pi}{c}\vec{j}_m, \quad \text{(Faraday)}$$

- Focus on the left-hand side: How big are the smallest charges?
- Charge (carrier) = a "thing" enclosed by a Gaussian surface
  - …which can then be shrunk down, right to the charge itself
  - …& detect the maximal extent of the minimal (elementary) charges
- E.g.: a Gaussian sphere can be shrunk radially to a ***point***.

Gauss's Laws

E&M Sources and Sinks in 4D

- As trivial as this may seem, (re)consider:

$$\Phi_E := \oint_{S_G} \mathrm{d}^2\vec{\sigma}\cdot\vec{E} = \oint_{S_G} \mathrm{d}^2\sigma^i\, E_i = \oint_{S_G} \mathrm{d}^2\sigma^{0i}\, F_{0i} = \oint_{S_G} \mathrm{d}x^\mu \mathrm{d}x^\nu\, \varepsilon_{\mu\nu}{}^{0i}\, F_{0i}.$$

- From the appearance of the Levi-Civita symbol $\varepsilon_{\mu\nu}{}^{\rho\sigma} := \varepsilon_{\mu\nu\kappa\lambda}\eta^{\kappa\rho}\eta^{\lambda\sigma}$
  - one of the indices contracted with $F_{\mu\nu}$ is time-like
  - one of the indices contracted with $F_{\mu\nu}$ is space-like
  - both of the indices contracted with $\mathrm{d}x^\mu \mathrm{d}x^\nu$ are space-like
  - they are all mutually orthogonal.
  - The last two are tangential to S_G—a static, spatial 2D surface
  - The third spatial direction is perpendicular to S_G
  - ...and in the direction of E_i , which emanates from/to the source/sink.
  - Thus, shrinking S_G along this third ("radial") spatial coordinate
  - ...identifies the source/sink as min(S_G):

point!

$$\dim(\rho_e) \geqslant \dim(\text{space}) - \dim(S_G) - \dim(\text{radius}) = 3 - 2 - 1 = 0.$$

Gauss's Laws

E&M Sources and Sinks in 4D

- For the magnetic field the story is a leeettle different:

$$\Phi_M := \oint_{S'_G} \mathrm{d}^2\sigma_i\, B^i = \oint_{S'_G} \mathrm{d}x^i \mathrm{d}x^j\, B^k\, \varepsilon_{0ijk} = \oint_{S'_G} \mathrm{d}x^j \mathrm{d}x^k\, F_{jk}$$

$$= \oint_{S'_G} \mathrm{d}x^\mu \mathrm{d}x^\nu\, \varepsilon_{\mu\nu 0i} \big(\tfrac{1}{2}\varepsilon^{0ijk}\, F_{jk}\big) = \oint_{S'_G} \mathrm{d}x^\mu \mathrm{d}x^\nu\, \varepsilon_{\mu\nu 0i} \big(*F^{0i}\big).$$

- This gives a dimension count similar to the one before:

$$\dim(\rho_m) \geqslant \dim(\text{space}) - \dim(S'_G) - \dim(\text{radius}) = 3 - 2 - 1 = 0.$$

$$\dim(\rho_e) \geqslant \dim(\text{space}) - \dim(S_G) - \dim(\text{radius}) = 3 - 2 - 1 = 0.$$

- But, note the subtle difference between the electric and magnetic Gaussian surfaces:

$$\Phi_E = \oint_{S_G} \mathrm{d}x^\mu \mathrm{d}x^\nu\, \varepsilon_{\mu\nu}{}^{0i}\, F_{0i} \qquad vs. \qquad \Phi_M = \oint_{S'_G} \mathrm{d}x^i \mathrm{d}x^j\, F_{ij}$$

- Both are 2-dimensional, but for *complementary* reasons.

Gauss's Laws

Other Gauge Sources and Sinks in n Dimensions

- To see how this *does* make a difference, consider *n*-dimensions:

$$\Phi_E := \oint_{S_G} \mathrm{d}x^{\mu_1}\cdots\mathrm{d}x^{\mu_{n-2}}\,\varepsilon_{\mu_1\cdots\mu_n}\eta^{\mu_{n-1}0}\eta^{\mu_n i}\,F_{0i},$$

$$\dim(\rho_e) \geqslant (n-1)-(n-2)-1 = 0$$

- as compared to

$$\Phi_M := \oint_{S_G} \mathrm{d}x^{\mu}\mathrm{d}x^{\nu}\,\tfrac{1}{(n-2)!}\varepsilon_{\mu\nu\rho_1\cdots\rho_{n-2}}\,(*F)^{\rho_1\cdots\rho_{n-2}},$$

$$\dim(\rho_m) \geqslant (n-1)-(2)-1 = n-4$$

- This follows from the dual identification:

$$E_i := F_{0i}, \qquad \text{but} \qquad B^{i_1\cdots i_{n-3}} := \varepsilon^{0i_1\cdots i_{n-3}jk}F_{jk}$$

rank-1 (w.r.t. rotations) rank-(n–3)

- ...and a rank-r tensor emanates "radially" from/to an (r–1)-dimensional source/sink.

Gauss's Laws

Other Gauge Sources and Sinks in n Dimensions

- In fact, this difference is even more dramatic when considering
  - rank-r gauge potentials: $A_{\mu_1\dots\mu_r}$, which define
  - a rank-r electric field $E_{i_1\dots i_r} := F_{0i_1\dots i_r}$
  - a rank-(n–r–2) magnetic field $B^{i_1\cdots i_{(n-r-2)}} := \varepsilon^{0i_1\cdots i_{(n-r-2)}j_1\cdots j_{r+1}} F_{j_1\dots j_{r+1}}$
  - where $F_{\mu_1\dots\mu_{r+1}} := \partial_{[\mu_1} A_{\mu_2\dots\mu_{r+1}]}$.
- The analogous computation and reasoning with the Gaussian surfaces produces the general result:
- Rank-r gauge potentials have

$$\dim(\rho_e) \geqslant \big((n-1)\big) - \big(n-(r+1)\big) - 1 = r-1: \quad E\text{-}(r-1)\text{-branes}$$

$$\dim(\rho_m) \geqslant \big((n-1)\big) - (r+1) - 1 = n-r-3: \quad M\text{-}(n-r-3)\text{-branes}$$

- where $0 \leqslant [p := (r-1)] \leqslant (n-4)$. $n \leq 11$, or 12
- E.g., in 6D, if electric charges are as little as point-like, magnetic charges cannot be smaller than membranes.

M-theory F-theory

Gauss's Laws

Oh, btw…

- All ‘branes have tensions that $\sim 10^{19}$ GeV and so a characteristic size $\sim 10^{-35}$ m.
- In an ordinary, closed $\sim 10^{-35}$ m string
  - amplitudes of the radial oscillations (“breathing mode”) are limited to $\sim 10^{-35}$ m by the $\sim 10^{19}$ GeV tension of the string
  - amplitudes of the transversal oscillations (and CM motion) are not limited to $\sim 10^{-35}$ m by the $\sim 10^{19}$ GeV tension of the string
  - The radial coordinate (“breathing mode”) is effectively “trapped” to be $\sim 10^{-35}$ m, but not so the remaining $n-1$ coordinates.
- A p-brane with a characteristic size $\sim 10^{-35}$ m can effectively “trap” p of the n dimensions and confine them to be compact
- $\Rightarrow$ a quite literally physical mechanism for compactifying p of the n dimensions, leaving $(n-p) \geq 4$ dimensions non-compact.

Nothing like this in any pre-string theory

Dirac's Dual Quantization

The Dirac-Wu-Yang Magnetic Monopole

- Recall the dually "symmetric" form of the Maxwell equations:

$$\partial_\mu F^{\mu\nu} = \frac{1}{4\pi\epsilon_0}\frac{4\pi}{c} j_e^\nu \qquad \partial_\mu \left(\tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\nu\rho}\right) = \frac{\mu_0}{4\pi}\frac{4\pi}{c} j_m^\sigma$$

- Now, the direct substitution of the standard

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

- ...produces

$$\Box A^\nu - \partial^\nu \partial\cdot A = \frac{1}{4\pi\epsilon_0}\frac{4\pi}{c} j_e^\nu \qquad 0 = \frac{\mu_0}{4\pi}\frac{4\pi}{c} j_m^\sigma$$

- since

$$\tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu \left(\partial_\nu A_\rho - \partial_\rho A_\nu\right) = \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\nu A_\rho - \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\rho A_\nu,$$

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\nu = \underbrace{\varepsilon^{\mu\nu\rho\sigma}\partial_\nu\partial_\mu}_{\mu\leftrightarrow\nu} = \varepsilon^{\nu\mu\rho\sigma}\partial_\mu\partial_\nu = -\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\nu = 0$$

- So, for nonzero magnetic charges to exist, $F_{\mu\nu} \neq \partial_\mu A_\nu - \partial_\nu A_\mu$ (*globally*)

Dirac's Dual Quantization

The Dirac-Wu-Yang Magnetic Monopole

- 1931, Dirac's realization: A_μ and $F_{\mu\nu}$ could be "singular".
- Desiring to produce a Coulomb-like

$$\vec{\nabla}\times\vec{A}\vec{B} = \frac{1}{4\pi}\frac{q_m}{r^2}\hat{r}$$

- model it with an infinitesimally thin, indefinitely long solenoid
- ...with the North pole at the origin, the South pole at $z = -\infty$.
- This "works" everywhere except along the negative z-axis
- ...which is physically inaccessible, being blocked by the solenoid.
- 44 years later, in 1975, T.T. Wu and C.Y. Yang found a loophole
- ...by means of gauge invariance:

$$A_\mu \quad \to \quad A'_\mu = A_\mu - c\,\partial_\mu\lambda$$

$$\vec{B} = \vec{\nabla}\times\vec{A} \quad \to \quad \vec{B}' = \vec{\nabla}\times\vec{A}' = \vec{\nabla}\times(\vec{A}' - \vec{\nabla}\lambda) = \vec{\nabla}\times\vec{A} = \vec{B}$$

Dirac's Dual Quantization

The Dirac-Wu-Yang Magnetic Monopole

- The Dirac-Wu-Yang magnetic monopole:

$$\vec{A}_N = \frac{q_m}{4\pi}\frac{x\,\hat{e}_y - y\,\hat{e}_x}{r(z+r)},$$

$$= -\frac{q_m}{4\pi}\frac{\cos(\theta)-1}{r\sin(\theta)}\,\hat{e}_\phi,$$

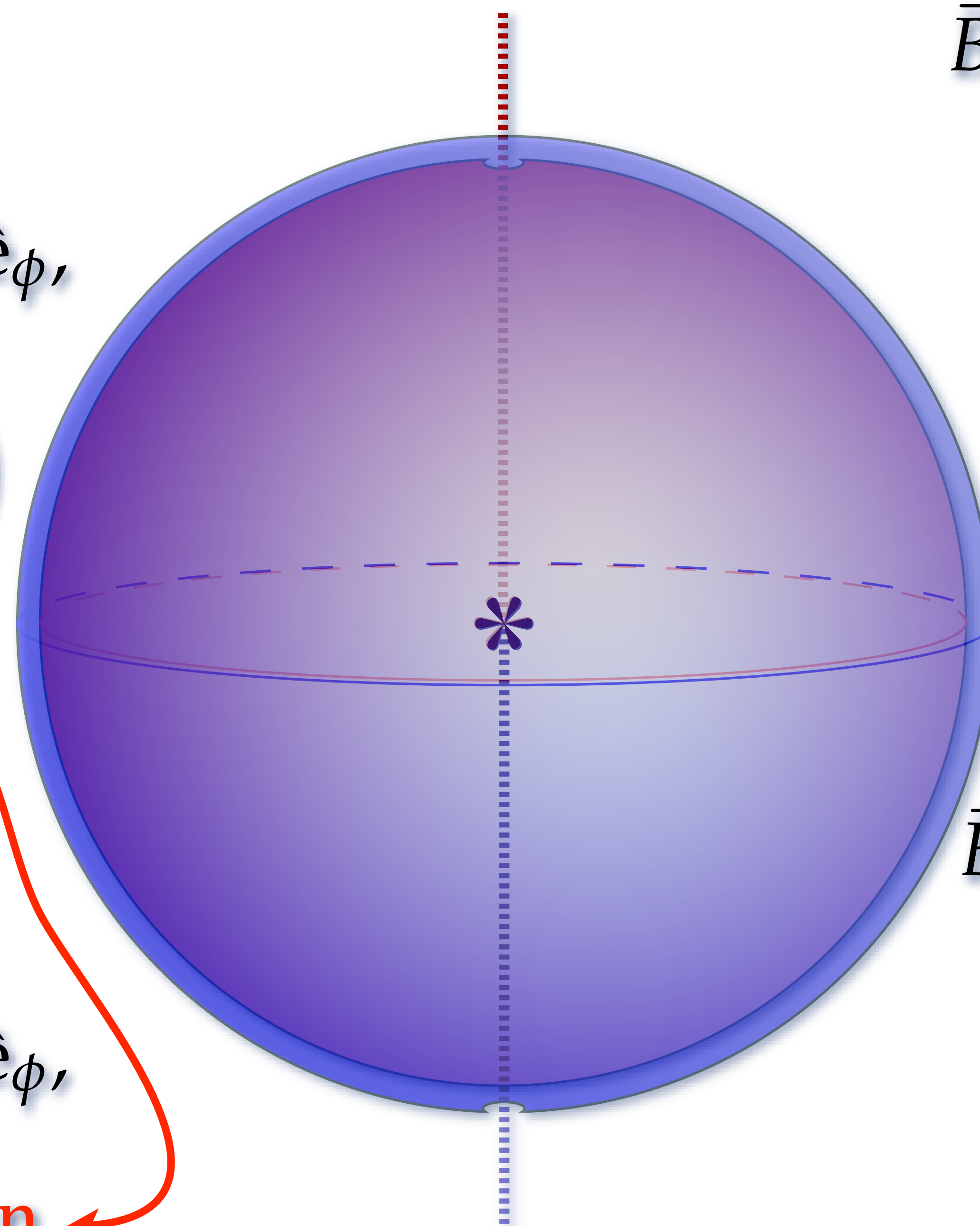
$$\vec{A}_S - \vec{A}_N = 2\frac{q_m}{4\pi}\left(\frac{y\hat{e}_x - x\hat{e}_y}{x^2+y^2}\right)$$

$$= -2\frac{q_m}{4\pi}\vec{\nabla}\phi$$

$$\vec{A}_S = \frac{q_m}{4\pi}\frac{x\,\hat{e}_y - y\,\hat{e}_x}{r(z-r)},$$

$$= -\frac{q_m}{4\pi}\frac{\cos(\theta)+1}{r\sin(\theta)}\,\hat{e}_\phi,$$

gauge transformation



$$\vec{B}_N := \vec{\nabla}\times\vec{A}_N,$$

$$= \frac{q_m}{4\pi}\frac{\vec{r}}{r^3}$$

except along $(0, 0, z \le 0)$

jointly, except @ (0, 0, 0)

$$\vec{B}_S := \vec{\nabla}\times\vec{A}_S$$

$$= \frac{q_m}{4\pi}\frac{\vec{r}}{r^3}$$

except along $(0, 0, z \ge 0)$

Dirac's Dual Quantization

The Dual Quantization of the E&M Charge

- The Dirac-Wu-Yang magnetic monopole:
- By picking two gauge-transformation related gauge potentials

$$\vec{A}_N \to \vec{A}_S = \vec{A}_N + \vec{\nabla}\lambda_{NS}, \quad \lambda_{NS}(\mathbf{x}) = -2\frac{q_m}{4\pi}\phi$$

$$\vec{B}_N \to \vec{B}_S = \vec{B}_N$$

- ...Wu & Yang obtained a magnetic monopole field that is well-defined everywhere except at the origin—where the charge is.
- The same gauge transformation acts on charged particles:

$$\Psi(\mathbf{x}) \to \Psi'(\mathbf{x}) = e^{iq_e\lambda_{NS}(\mathbf{x})/\hbar}\,\Psi(\mathbf{x}) = e^{-i\,\frac{q_m q_e}{2\pi\hbar}\phi}\,\Psi(\mathbf{x})$$

- which must remain single-valued. Whence

$$\frac{q_e q_m}{2\pi\hbar} \overset{!}{=} n \in \mathbb{Z}, \qquad i.e., \qquad q_m \overset{!}{=} n\left(\frac{2\pi\hbar}{q_e}\right)$$

- is the Dirac (dual) quantization of E&M charges.

Dirac's Dual Quantization

The Dual Quantization of the E&M Charge

- This argument can be recast further:

$$\int_1^2 \mathrm{d}\vec{r}\cdot\vec{A}_N = \int_1^2 \mathrm{d}\vec{r}\cdot\vec{A}_S + \Big[\int_1^2 \mathrm{d}\vec{r}\cdot\vec{\nabla}\lambda_{NS} = \lambda_{NS}(\vec{r}_2) - \lambda_{NS}(\vec{r}_1)\Big]$$

- *i.e.*,

$$\int_1^2 \mathrm{d}\vec{r}\cdot\vec{A}_N - \int_1^2 \mathrm{d}\vec{r}\cdot\vec{A}_S = \lambda_{NS}(\vec{r}_2) - \lambda_{NS}(\vec{r}_1)$$

- so

$$e^{iq_e \int_{C_1} \mathrm{d}\vec{r}\cdot\vec{A} - q_e \int_{C_2} \mathrm{d}\vec{r}\cdot\vec{A}} = e^{iq_e \oint_{C_1 - C_2} \mathrm{d}\vec{r}\cdot\vec{A}} \stackrel{!}{=} e^{2\pi i\, n} = 1$$

- where C_1 and C_2, having the same endpoints, form a closed contour:

$$2\pi i\, n \stackrel{!}{=} iq_e \int_{C_1} \mathrm{d}\vec{r}\cdot\vec{A} - iq_e \int_{C_2} \mathrm{d}\vec{r}\cdot\vec{A} = iq_e \oint_{(C_1 - C_2) = \partial S} \mathrm{d}\vec{r}\cdot\vec{A}$$

$$= iq_e \int_S \mathrm{d}^2\vec{\sigma}\cdot\vec{B},$$

- Notice that the same condition induces the Aharonov-Bohm & Aharonov-Casher effect. (The ABCD effect.)

Dirac's Dual Quantization

Two Other Arguments

- 1965, Alfred S. Goldhaber: the Lorentz force $\vec{F}_L = q_e\,\vec{v} \times \vec{B}$,
- acts on a particle approaching a magnetic charge.
- Let the initial velocity be in the z-direction; the magnetic field in the (x,z)-plane causes a deflection in the y-direction:

$$(\triangle\vec{p})_y \approx \int_{-\infty}^{+\infty} \mathrm{d}t\;(\vec{F}_L)_y = \frac{q_e v q_m b}{4\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}t}{(b^2+v^2t^2)^{3/2}} = \frac{q_e\,q_m}{2\pi b},$$

$$(\triangle\vec{L})_z = b(\triangle\vec{p})_y = \frac{q_e q_m}{2\pi}, \quad b := r\big|\sin\measuredangle(\vec{B},\vec{v})\big|.$$

- Quantization of the *increments* of angular momentum in integral multiples of $\hbar$ reproduces Dirac's dual quantization.

- Even faster, J.J. Thomson, 1904: The total angular momentum of a point-like electric and magnetic charge a distance R apart with the EM field is

$$\vec{L}_{EM} = \frac{q_e q_m}{4\pi}\frac{\vec{R}}{R}.$$

Quantizing its magnitude in half-integral multiples of $\hbar$ recovers Dirac's dual quantization.

Dualities and Dual Worldviews

Parameter Spaces for Points and for Strings

- Introduce a simplifying notation:

$$\mathbf{d} := \mathrm{d}x^{\mu}\partial_{\mu}, \qquad \mathbf{A}_{(r)} := \mathrm{d}x^{\mu_1}\cdots \mathrm{d}x^{\mu_r} A_{\mu_1\cdots\mu_r}(\mathbf{x}),$$

$$\mathbf{F}_{(r+1)} := \mathbf{d}\mathbf{A}_{(r)} := \mathrm{d}x^{\mu_1}\cdots \mathrm{d}x^{\mu_{r+1}}\big[F_{\mu_1\cdots\mu_{r+1}}(\mathbf{x}) := \big(\partial_{[\mu_1}A_{\mu_2\cdots\mu_{r+1}]}(\mathbf{x})\big)\big].$$

- Then

$$q_e^{(r)}\int_S \mathbf{F}_{(r+1)} = q_e^{(r)}\oint_{\partial S}\mathbf{A}_{(r)} \stackrel{!}{=} 2\pi\, n_{(r)} \in 2\pi\mathbb{Z}, \quad \forall S, \quad \dim(S) = (r{+}1)$$

$$q_m^{(n-r-2)}\int_{S'}(*\mathbf{F})_{(n-r-1)} = q_m^{(n-r-2)}\oint_{\partial S'}(\mathbf{A}^*)_{(n-r-2)} \stackrel{!}{=} 2\pi\, n_{(n-r-2)} \in 2\pi\mathbb{Z}$$

- When $\mathscr{X} = \mathscr{X}' \times \mathscr{Y}$ (dim = 10, 4, 6)
- …there are two pairs of fields: using the 10,- 4- and 6-d ε's

$$\mathbf{F}_{(r+1)}, \qquad (*_{\mathscr{X}}\mathbf{F})_{(10-r-1)}, \qquad (*_{\mathscr{X}'}\mathbf{F})_{(4-r-1)}, \qquad (*_{\mathscr{Y}}\mathbf{F})_{(6-r-1)},$$

- and *independent* dual quantization within $\mathscr{X}$, $\mathscr{X}'$ and/or $\mathscr{Y}$.

Dualities and Dual Worldviews

Parameter Spaces for Points and for Strings

- Within (M- and F-extended) "string theory," there exist more restrictions than in pointillist theories.
  - Critical dimension
  - Anomaly cancellations from
    - worldsheet perspective
    - target spacetime perspective
    - higher perspectives…
  - Multiple Dirac (dual) quantization
  - Parameter space limitation
    - Field dynamics
    - determines the metric on the field space
    - determines the volume of the field space

$$\mathrm{Vol}_S(C) = \left| \int_{2\pi\ell_S}^{\infty} \frac{\mathrm{d}r}{r} \right|$$

Pointillist circle compactification

- Circumference $C \in [0, \infty)$
- $C = \infty$ is "decompactification"
- $C = 0$ is a wholly new theory (with one fewer dimension)
- $\mathrm{Vol}(C) = \infty$

$$[0, \infty) > [\ell_S, \infty)$$

Stringy circle compactification

- Circumference $C \in [\ell_S, \infty)$
- $C = \infty$ is "decompactification"
- No $C = 0$, as this is $\simeq\infty$.
- $\mathrm{Vol}(C) = \infty$, only logarithmically

Dualities and Dual Worldviews

Parameter Spaces for Points and for Strings

- Even better example is the case of moduli spaces, esp. those of Calabi-Yau n-folds
- Various dualities reduce the parameter spaces to an effective (complete and non-redundant) moduli space
- The physically motivated metric stems from the (variable) "kinetic" term in the ("sigma-model") action

$$\int d^4x\, \eta^{\mu\nu}\, (\partial_\mu \phi^{\bar{a}}(x))\, \mathcal{G}_{\bar{a}b}(\phi,\bar{\phi})\, (\partial_\nu \phi^{b}(x)) + \ldots$$

  - where ϕ^a are complex scalar fields in the 4D field theory, but take values in the moduli space of the compact 6D CY 3-fold
  - and where the metric is $\mathcal{G}_{\bar{a}b}(\phi,\bar{\phi})$
  - This (so-called Zamolodchikov) metric coincides with the (mathematically) canonical, Weil-Petersson metric
  - in which the Vol(moduli space) is finite, as are distances within it.

Andrei Todorov, 2007

P.Candelas, P.Green, T.H., 1990

Dualities and Dual Worldviews

The Gauge-Completeness Conjecture

- Honing the observation that (*M*- and *F*-extended) “string theory” has more restrictions than pointillist theories:
  - landscape of all completely quantum & (general & gauge) relativistically consistent models
  - within a swam of incompletely consistent models.
- Our Nature is believed to be describable by one of the models within the landscape.
- Fully completed (*M*- and *F*-extended) “string theory” will by definition be a nonempty subset (perhaps all of) the landscape.
- Conjecture:

> In consistent quantum models with gravity, (**1**) all symmetries are gauged, and (**2**) all sources/sinks for all gauge symmetries are included & satisfy the appropriate dual charge quantization.

A "Theory" of More Than Everything

A Theoretical (Axiomatic) System, Really

- String theory was developed and applied to hadrons
  - owing to the gravity/gauge (generalizing the AdS/CFT) duality, string theory is now also used to revisit gauge theory and QCD
- It is now used in studies of condensed matter physics
  - where condensed matter experiments model cosmology
  - where methods of supersymmetry, string theory, gauge theory and the holography principle are applied to condensed matter
- It is of course also studied as the candidate for the fundamental theory of the elementary "*particles*" and their interactions
- It is thus more than "a (single) theory," a theoretical *system*
  - Contains a telescoping hierarchy of field theories
  - Has a continuum/discretuum of possible models
  - Somewhat like Classical Mechanics, Electricity & Magnetism, Statistical Mechanics; Quantum Theory, Relativity Theory, Gauge Theory… …except "string theory" ***contains*** them all!

Thanks!

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