(Fundamental) Physics of Elementary Particles

Supersymmetric Field Theory; Supersymmetry Breaking; Superfields in 3+1-Dimensional Spacetime; Constrained and Gauge Superfields

Tristan Hübsch
Department of Physics and Astronomy
Howard University, Washington DC
Prirodno-Matematički Fakultet
Univerzitet u Novom Sadu

Fundamental Physics of Elementary Particles

Program

- Supersymmetric Field Theory
  - Ideas and Theorems
  - General Properties
  - Motivations
- Supersymmetry Breaking
- Superfields in 3+1-Dimensional Spacetime
  - Superalgebra
  - Superspace
  - Superfields
- Constrained and Gauge Superfields
  - Superconstrained Superfields
  - The Chiral Ring and Exact Results
  - Supersymmetric Electrodynamics

Supersymmetric Field Theory

Ideas and Theorems

- Before the November 1974 acceptance of the quark model
- ...attempts to ascertain general restriction particle physics, without detailed knowledge of the dynamics
- Also, attempts to combine
  - spacetime symmetry, such as rotation/spin $SU(2)$
  - *internal* symmetries, such as $SU(3)_f$
- Non-relativistic models using $SU(6) \supset SU(2) \times SU(3)_f$ work well
- But, no relativistic generalization could be constructed
- In 1965, L. O'Raifeartaigh proved that

  The Hilbert space of states of a particle with a finite nonzero mass is Poincaré-invariant

- P. Roman & C.J. Koh:

  Distinct particles and states transformed into each other by an internal group have the same Lorentz-invariant mass.

Supersymmetric Field Theory

Ideas and Theorems

- In 1967, S. Coleman & J. Mandula

In any interactive relativistic model of particles with finite masses, the only permissible symmetries must form the Poincaré group and a Lie group that commutes with the Poincaré group

- That is, *internal* symmetry (such as color, isospin, $SU(3)_f$ …) G
  - must commute with the Poincaré group
  - cannot change any of the eigenvalues of the Poincaré generators
  - …such as spin, spin projection, mass $m := \sqrt{\mathrm{p}\cdot\mathrm{p}}$
- As spin (eigenvalue of rotation generators) cannot be changed
- particles of different spin cannot be in the same G-multiplet

Supersymmetric Field Theory

Ideas and Theorems

- In 1975, R. Haag, J. Łopuszański & M. Sohnius
- …noticed an implicit assumption in the Coleman-Mandula proof
- ***Everyone*** assumed that all transformation operators
  - …are of the form ***transformation*** = $e^{i(\textbf{parameter})\cdot(\textbf{generator})}$,
  - …so generators form algebraic structures *via* the commutator
  - …and produce Lie algebras (as per a theorem by Wigner, 1930's)
- But, some generators themselves *could* be fermionic
  - …and need the anticommutator binary operation in superalgebras:

$$[B_1, B_2] = B_3 \qquad [B_1, F_2] = F_3 \qquad \{F_1, F_2\} = F_3$$

  - $\mathbb{Z}_2$-graded algebras: $|B_i| = 0$ (even) and $|F_i| = 1$ (odd).
- …and can "link" the algebra (*Poincaré*)⊕(*"internal"*) nontrivially
- …extending it into a proper ***superalgebra***

Supersymmetric Field Theory

General Properties

- The Poincaré algebra $\mathfrak{po}(1,3) = \mathfrak{spin}(1,3) \ltimes \mathfrak{tr}(\mathbb{R}^{1,3})$
  $\qquad\qquad\qquad\qquad L \qquad\qquad P$
  - consists of:

$$[L, L'] = L,'' \qquad [L, P] = P', \qquad [P, P'] = 0.$$

  - Is extended:

$$\{Q, Q'\} = P \oplus Z, \quad [L, Q] = \tfrac{1}{2} Q', \quad [P, Q] = 0,$$

$$[Z, Z'] = Z,'' \qquad [L, Z] = 0, \qquad [Z, P] = 0, \quad [Z, Q] = 0.$$

  - by the ***supercharges*** Q and the ***central charges*** Z.
  - The P have spin 1, the Q have spin ½, the Z have spin 0.
  - The Q's, exclusively, transform $|boson\rangle \leftrightarrow |fermion\rangle$
  - Without inserting dimensional parameters into the algebra

$$[Q] = \sqrt{\frac{ML}{T}}, \qquad [P] = [Z] = \frac{ML}{T}, \qquad [L] = 0.$$

  - Also,

$$P_0 = -H/c.$$

Supersymmetric Field Theory

Motivations

- Quantumness:
  - The only universal quality in Nature that stabilizes atoms
    - Quantized angular momentum obstructs radiation loss & instability
    - Not the original motivation (1913, Bohr: recovery of line spectra)
  - Unifies concepts (mnemonic imagery) of waves & particles
- Gauge principle:
  - Links symmetries with interactions
    - Unobservable-ness of wave-function (matrix-generalized) phases
    - Arbitrary spacetime variability of these phases ⇒ gauge potential
    - Gauge potential & gauge fields ⇒ gauge interaction (measurable)
  - Unifies concepts (mnemonic imagery) of particles, waves & fields
  - Puzzle:
    - How can ***unobservable*** (matrix-generalized) phases
    - produce ***observable*** interactions?!

Supersymmetric Field Theory

Motivations

- Special relativity:
  - Links symmetries & conservation laws
  - Unifies space+time, energy+momentum, rotations+boosts
- General relativity:
  - Links symmetries & gravity & spacetime geometry
    - Unobservable-ness of coordinate choices
    - Arbitrary spacetime variability of coordinates ⇒ Christoffel symbol
    - Christoffel symbol & Riemann curvature ⇒ gravitation: observable force
    - gravitation ⇔ spacetime curvature: no force, just free-fall—albeit curved
  - This "geometrization" recovers the un-observability
    of the original coordinate transformation as a symmetry
  - and the gravitational interaction as its consequence
  - This "geometrization" can be extended to all of the gauge interactions
    - by extending the spacetime → (spacetime, "phases") where particles move

fiber bundle

Supersymmetric Field Theory

Motivations

- Supersymmetry:
  - The only universal quality in Nature that stabilizes the vacuum
    - The minimum energy is zero if and only if the system is supersymmetric
    - The minimum energy is positive if supersymmetry is spontaneously broken
    - If the system includes general relativity, energy is not well-defined
  - The only unification of fermions (matter) & bosons (interactions)
  - Technical advantages:
    - Significantly lessens (or even eliminates) need for renormalization
    - Prevents mixing of characteristic energies
    - Preserves unnaturally small/large ratios

$$\frac{m_{\nu_e}}{M_P} \lesssim 10^{-28}, \quad \frac{m_e}{M_P} \sim 10^{-23}, \quad \frac{m_u}{M_P} \sim 10^{-22}$$

$$M_P = \sqrt{\frac{\hbar c}{G_N}}$$

- The true, complete theory is presumably and hopefully, simply finite.

Supersymmetric Field Theory

Motivations

- Vacuum energy:
- Consider a free scalar field

$$\mathscr{L}_{\text{KGB}} = \tfrac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - \tfrac{1}{2}\left(\tfrac{mc}{\hbar}\right)^2\phi^2 = \tfrac{1}{2c^2}\dot{\phi}^2 - \tfrac{1}{2}\left[\vec{\nabla}^2 + \left(\tfrac{mc}{\hbar}\right)^2\right]\phi^2.$$

- The Euler-Lagrange equation of motion is

$$\left[\tfrac{1}{c^2}\partial_t^2 - \vec{\nabla}^2 + \left(\tfrac{mc}{\hbar}\right)^2\right]\phi(\mathbf{x}) = 0,$$

- the Fourier transform of which gives

$$\phi(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3\vec{k}\;\phi_{\vec{k}}(\mathbf{x}), \qquad \phi_{\vec{k}}(\mathbf{x}) := f_{\vec{k}}(t)\,e^{i\vec{k}\cdot\vec{r}},$$

$$\left[\partial_t^2 + \left(\vec{k}^2 c^2 + \tfrac{m^2c^4}{\hbar^2}\right)\right] f_{\vec{k}}(t) = 0.$$

- A bunch of LHO's:

$$E_{n,\vec{k}} = E_{\vec{k}}(n + \tfrac{1}{2}), \qquad E_{\vec{k}} := \hbar c\sqrt{\vec{k}^2 + \tfrac{m^2c^2}{\hbar}} = \sqrt{(\hbar\vec{k})^2c^2 + m^2c^4}$$

Supersymmetric Field Theory

Motivations

- The vacuum energy

$$E_{\text{vacuum}} = \frac{1}{2}\int \mathrm{d}^3\vec{k}\, E_{\vec{k}} = 2\pi \int_0^\infty k^2 \mathrm{d}k\, \sqrt{\hbar^2 k^2 c^2 + m^2 c^4}$$

- diverges $\sim k^4$ as $k \to \infty$
- For the free electromagnetic field, $m = 0$ and so $E_{\vec{k}} = |\hbar\vec{k}|c$
- ...and the vacuum energy divergence remains.
- Remember the supersymmetric LHO?

$$E_{n,\nu} = \hbar\left[\omega(n+\tfrac{1}{2}) + \widetilde{\omega}(\nu - \tfrac{1}{2})\right]. \qquad E_{0,0} = \tfrac{1}{2}\hbar(\omega - \widetilde{\omega}).$$

$\widetilde{\omega} \to \omega$

$$E_{n,\nu} = \hbar\omega(n+\nu). \qquad E_{0,0} = 0.$$

- For every bosonic mode, a fermionic mode must be added
- ...that is, every bosonic field needs a fermionic superpartner.
- O'Raifeartaigh: (matter) fermions $\not\leftrightarrow$ (interaction) bosons

Particle doubling

Supersymmetry Breaking

A Few Small Bites of a Tough Cookie

- Dimensionally reduce to the worldline

$$\{\, Q^{\dagger i}, Q_j \,\} = \delta^i_j\, H, \quad \sum_i \{Q^{\dagger i}, Q_i\} = N H, \quad \mathrm{Tr}[\delta^i_j] = N,$$

- so

$$0 = \langle\Omega|H|\Omega\rangle = \Big\langle \Omega \Big| \tfrac{1}{N} \sum_i \{Q^{\dagger i}, Q_i\} \Big| \Omega \Big\rangle$$

$$= \tfrac{1}{N} \sum_i \Big\{ \big| Q_i|\Omega\rangle \big|^2 + \big| Q^{\dagger i}|\Omega\rangle \big|^2 \Big\},$$

- ...which is a sum of non-negative contributions.
- Thus

$$H|\Omega\rangle = 0 \qquad \Leftrightarrow \qquad Q_i|\Omega\rangle = 0 = Q^{\dagger i}|\Omega\rangle.$$

- In turn,

$$U_{\epsilon,\bar{\epsilon}}|\Omega\rangle = |\Omega\rangle, \qquad U_{\epsilon,\bar{\epsilon}} := \exp\{-i(\epsilon\cdot Q + \epsilon^\dagger \cdot Q^\dagger)\}$$

- Supersymmetry-invariant states have zero energy.

& are global ground-states.

Supersymmetry Breaking

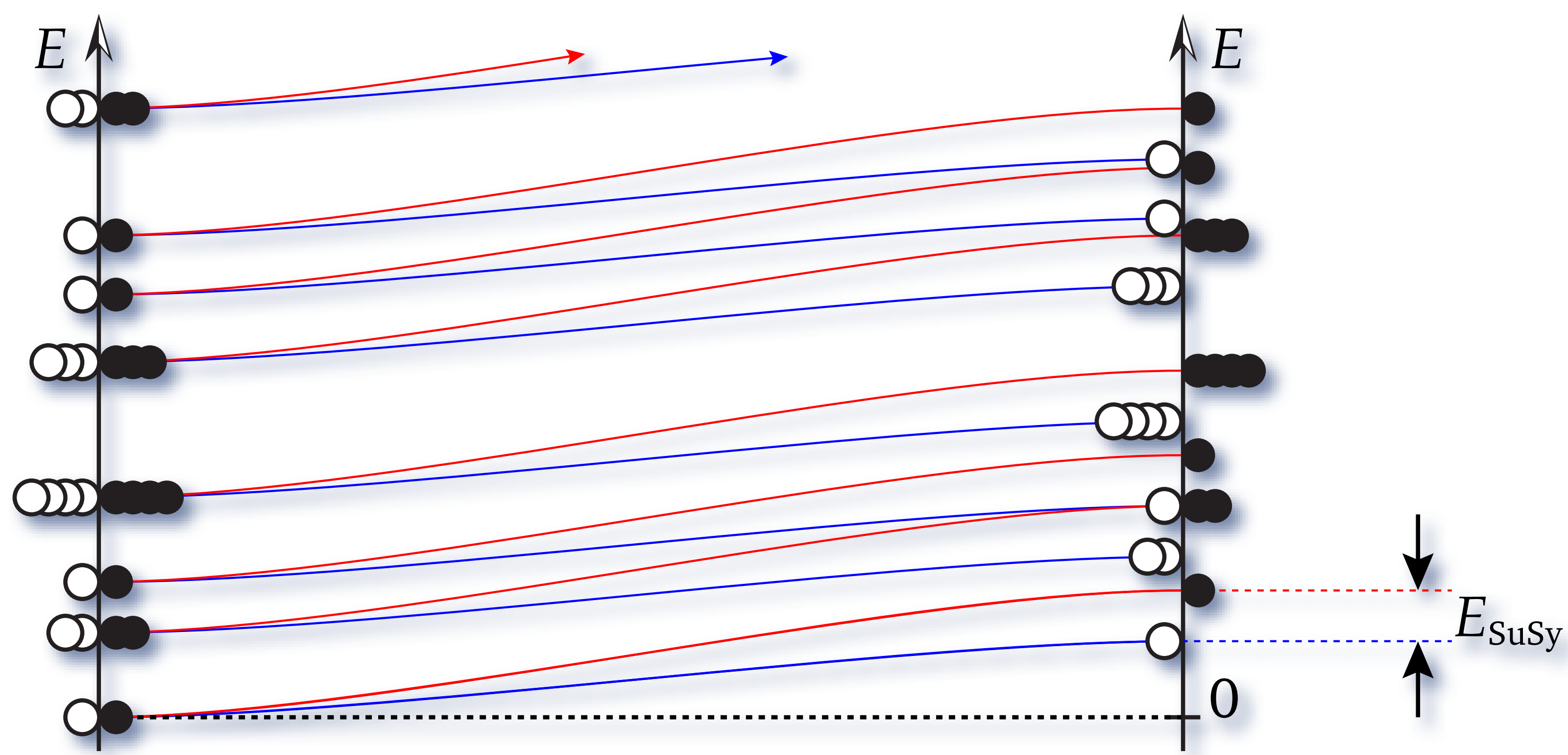
A Few Small Bites of a Tough Cookie

- There can exist (even continuously) many ground states

$$\iota_W := \chi_E(\mathscr{V}_B) - \chi_E(\mathscr{V}_F)$$

space of bosonic ground states   space of fermionic ground states

- The Euler characteristic reduces to the count for discrete points.
- Non-ground states are boson-fermion (degenerate) pairs



Supersymmetry Breaking

A Few Small Bites of a Tough Cookie

- Spontaneous supersymmetry breaking
  - If the Hamilton action is supersymmetry-invariant
  - but there exists no supersymmetric ground state
  - Discovered by O'Raifeartaigh
    - requires ≥ 3 complex (spin-0 | spin-½) field pairs
    - a specific choice of the potential (masses & interactions)
- Mediated supersymmetry breaking
  - A dynamical effect for a sub-sector in the model
  - …induces supersymmetry-breaking
- Explicit supersymmetry breaking
  - Adding supersymmetry-breaking terms "by hand"
  - Free field Lagrangians w/equal # & mass are supersymmetric
  - so, interactions may be considered as breaking supersymmetry

Superfields in 3+1-Dimensional Spacetime

Superalgebra

- The complex Weyl basis

$$\Psi \equiv \Psi_+ + \Psi_- = (\boldsymbol{\gamma}_+ \Psi) + (\boldsymbol{\gamma}_- \Psi),$$

$$\Psi_+ \mapsto \begin{bmatrix} \psi_\alpha \\ 0 \end{bmatrix} \quad \text{and} \quad \Psi_- \mapsto \begin{bmatrix} 0 \\ \overline{\chi}_{\dot\alpha} \end{bmatrix}, \quad \alpha, \dot\alpha = 1, 2.$$

- The Poincaré algebra:

$$P_\mu = \tfrac{\hbar}{i}\partial_\mu, \quad \text{and} \quad L_{\mu\nu} := \tfrac{\hbar}{i}(\eta_{\mu\rho}x^\rho\partial_\nu - \eta_{\nu\rho}x^\rho\partial_\mu),$$

- extended by supercharges:

$$\{\, Q_\alpha\, ,\, \overline{Q}_{\dot\alpha} \,\} = -2\,\sigma^\mu_{\alpha\dot\alpha}\, P_\mu, \qquad [\, L_{\mu\nu}\, ,\, Q_\alpha \,] = i\hbar(\boldsymbol{\sigma}_{\mu\nu})_\alpha{}^\beta Q_\beta,$$

$$[\, L_{\mu\nu}\, ,\, P_\rho \,] = i\hbar(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \qquad [\, L_{\mu\nu}\, ,\, \overline{Q}_{\dot\alpha} \,] = i\hbar(\overline{\boldsymbol{\sigma}}_{\mu\nu})_{\dot\alpha}{}^{\dot\beta}\overline{Q}_{\dot\beta},$$

$$[\, L_{\mu\nu}\, ,\, L_{\rho\sigma} \,] = i\hbar(\eta_{\mu\rho}L_{\nu\sigma} - \eta_{\mu\sigma}L_{\nu\rho} + \eta_{\nu\sigma}L_{\mu\rho} - \eta_{\nu\rho}L_{\mu\sigma}),$$

- with all other (anti)commutators vanishing.

Superfields in 3+1-Dimensional Spacetime

Superspace

- 1974, A. Salam & J. Strathdee introduced superspace
- Those who use it, those who consider it a bookkeeping artifice.
- 2008 (34 years later): it is inevitable. $\theta^i := \delta^{ij}\,[\,Q_j, t\,] \neq 0$

$$\mathbf{x} \;\mapsto\; (x^\mu; \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) = (ct, x^1, x^2, x^3; \theta^1, \theta^2, \bar{\theta}^{\dot{1}}, \bar{\theta}^{\dot{2}}),$$

$$\{\theta^\alpha, \theta^\beta\} = 0 = \{\theta^\alpha, \bar{\theta}^{\dot{\alpha}}\}.$$

$$\partial_\alpha := \frac{\partial}{\partial\theta^\alpha}, \quad \bar{\partial}_{\dot{\alpha}} := \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}, \qquad \{\partial_\alpha, \partial_\beta\} = \{\partial_\alpha, \bar{\partial}_{\dot{\beta}}\} = \{\bar{\partial}_{\dot{\alpha}}, \bar{\partial}_{\dot{\beta}}\} = 0.$$

- Then

$$Q_\alpha := i\partial_\alpha + \hbar\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \qquad \text{and} \qquad \bar{Q}_{\dot{\alpha}} := i\bar{\partial}_{\dot{\alpha}} + \hbar\sigma^\mu_{\alpha\dot{\alpha}}\theta^\alpha\partial_\mu$$

$$\{\, Q_\alpha \,,\, \bar{Q}_{\dot{\alpha}} \,\} = -2\,\sigma^\mu_{\alpha\dot{\alpha}}\, P_\mu,$$

- ...& are spin-½ generators of supersymmetry.

Superfields in 3+1-Dimensional Spacetime

Superspace

$$Q_\alpha := i\partial_\alpha + \hbar\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu, \qquad \text{and} \qquad \bar Q_{\dot\alpha} := i\bar\partial_{\dot\alpha} + \hbar\sigma^\mu_{\alpha\dot\alpha}\theta^\alpha\partial_\mu$$

- Interestingly,

$$D_\alpha := \partial_\alpha + i\hbar\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu, \qquad \text{and} \qquad \bar D_{\dot\alpha} := \bar\partial_{\dot\alpha} + i\hbar\sigma^\mu_{\alpha\dot\alpha}\theta^\alpha\partial_\mu$$

$$\{ D_\alpha , \bar D_{\dot\alpha} \} = -2\sigma^\mu_{\alpha\dot\alpha}P_\mu = 2i\hbar\sigma^\mu_{\alpha\dot\alpha}\partial_\mu,$$

- just like

$$\{ Q_\alpha , \bar Q_{\dot\alpha} \} = -2\,\sigma^\mu_{\alpha\dot\alpha}\,P_\mu,$$

- and

$$\left.\begin{array}{lcl} \{ D_\alpha , Q_\beta \} & = 0 = & \{ D_\alpha , \bar Q_{\dot\beta} \} \\ \{ \bar D_{\dot\alpha} , Q_\beta \} & = 0 = & \{ \bar D_{\dot\alpha} , \bar Q_{\dot\beta} \} \end{array}\right\} \Leftrightarrow \left\{\begin{array}{ll} U^{-1}_{\epsilon,\bar\epsilon} D_\alpha U_{\epsilon,\bar\epsilon} & = D_\alpha, \\ U^{-1}_{\epsilon,\bar\epsilon} \bar D_{\dot\alpha} U_{\epsilon,\bar\epsilon} & = \bar D_{\dot\alpha}. \end{array}\right.$$

- These are "~~covariant~~ *supersymmetry-invariant* superderivatives."
- Also:

$$-iQ_\alpha = D_\alpha - 2i\hbar\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu, \qquad \text{and} \qquad -i\bar Q_{\dot\alpha} = \bar D_\alpha - 2i\hbar\sigma^\mu_{\alpha\dot\alpha}\theta^\alpha\partial_\mu.$$

Superfields in 3+1-Dimensional Spacetime

Superfields

- Superfields = functions over superspace

$$\{\theta^\alpha,\theta^\beta\}=0 \quad \overset{\alpha=\beta}{\Longrightarrow} \quad 0=\{\theta^\alpha,\theta^\alpha\}=2(\theta^\alpha)^2 \quad \Rightarrow \quad (\theta^\alpha)^2=0,\ \ \alpha=1,2;$$

$$\{\bar\theta^{\dot\alpha},\bar\theta^{\dot\beta}\}=0 \quad \overset{\dot\alpha=\dot\beta}{\Longrightarrow} \quad 0=\{\bar\theta^{\dot\alpha},\bar\theta^{\dot\alpha}\}=2(\bar\theta^{\dot\alpha})^2 \quad \Rightarrow \quad (\bar\theta^{\dot\alpha})^2=0,\ \ \dot\alpha=1,2;$$

- so power-series terminate:

$$\mathbb{F}(\mathbf{x};\theta,\bar\theta)=\phi(\mathbf{x})+\theta^\alpha\psi_\alpha(\mathbf{x})+\bar\theta^{\dot\alpha}\bar\chi_{\dot\alpha}(\mathbf{x})+\cdots+\theta^2\bar\theta^2\mathcal{F}(\mathbf{x})$$

- where

$$\theta^2:=\tfrac{1}{2}\varepsilon_{\alpha\beta}\theta^\alpha\theta^\beta \quad \text{and} \quad \bar\theta^2:=\tfrac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta}$$

- The expansion “coefficients” are ordinary functions over ordinary spacetime, a.k.a. “component fields”
- They alternate in spin-statistics: boson/fermion/boson/…
- Some authors prefer to manipulate the power-series…

Superfields in 3+1-Dimensional Spacetime

Superfields

- Using the supersymmetry-invariance of superderivatives,

$$
\begin{aligned}
\phi(\mathrm{x}) &:= \mathbb{F}(\mathrm{x};\theta,\bar\theta)\big|; \qquad\qquad (\,X\,)\big| := \lim_{\theta,\bar\theta\to 0}(\,X\,) \\
\psi_\alpha(\mathrm{x}) &:= \big[D_\alpha\mathbb{F}(\mathrm{x};\theta,\bar\theta)\big]\big|; \\
\bar\chi_{\dot\alpha}(\mathrm{x}) &:= \big[\bar D_{\dot\alpha}\mathbb{F}(\mathrm{x};\theta,\bar\theta)\big]\big|; \\
F(\mathrm{x}) &:= -\tfrac{1}{4}\big[D^2\mathbb{F}(\mathrm{x};\theta,\bar\theta)\big]\big|; \\
V_{\alpha\dot\alpha}(\mathrm{x}) &:= -\tfrac{1}{2}\big[[D_\alpha,\bar D_{\dot\alpha}]\mathbb{F}(\mathrm{x};\theta,\bar\theta)\big]\big|, \qquad V_\mu := \tfrac{1}{2}\bar\sigma_\mu^{\dot\alpha\alpha}V_{\alpha\dot\alpha}; \\
G(\mathrm{x}) &:= -\tfrac{1}{4}\big[\bar D^2\mathbb{F}(\mathrm{x};\theta,\bar\theta)\big]\big|; \\
\lambda_\alpha(\mathrm{x}) &:= -\tfrac{1}{4}\big[\bar D^2 D_\alpha\mathbb{F}(\mathrm{x};\theta,\bar\theta)\big]\big|; \\
\bar\kappa_{\dot\alpha}(\mathrm{x}) &:= -\tfrac{1}{4}\big[D^2\bar D_{\dot\alpha}\mathbb{F}(\mathrm{x};\theta,\bar\theta)\big]\big|; \\
\mathcal{F}(\mathrm{x}) &:= \tfrac{1}{32}\big[(D^2\bar D^2+\bar D^2 D^2)\mathbb{F}(\mathrm{x};\theta,\bar\theta)\big]\big|
\end{aligned}
$$

- Supersymmetry-invariant definitions of component fields.

  No such thing for symmetries!

Superfields in 3+1-Dimensional Spacetime

Superfields

- Since all component field definitions involve setting $\theta, \bar{\theta} \to 0$
- and

$$-iQ_\alpha = D_\alpha - 2i\hbar\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu, \qquad \text{and} \qquad -i\bar{Q}_{\dot{\alpha}} = \bar{D}_\alpha - 2i\hbar\sigma^\mu_{\alpha\dot{\alpha}}\theta^\alpha\partial_\mu$$

- the superderivatives can be used to compute supersymmetry transformations.
- However, $Q_\alpha(\mathbb{F}) = \mathbb{F}\overleftarrow{Q_\alpha} = +(Q_\alpha\mathbb{F})$,
- implies $Q_\alpha(D_\beta\mathbb{F}) = (D_\beta\mathbb{F})\overleftarrow{Q_\alpha} = -(Q_\alpha \circ D_\beta\,\mathbb{F}))$
- So

$$\delta_Q(\epsilon)\phi = (\epsilon^\alpha D_\alpha + \bar{\epsilon}^{\dot{\alpha}}\bar{D}_{\dot{\alpha}})\mathbb{F}\big| = \epsilon^\alpha\psi_\alpha + \bar{\epsilon}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}};$$

$$\delta_Q(\epsilon)\psi_\alpha = (\epsilon\cdot D + \bar{\epsilon}\cdot\bar{D})(D_\alpha\mathbb{F})\big| = \tfrac{1}{2}\epsilon^\beta\varepsilon_{\beta\alpha}D^2\mathbb{F}\big| + \bar{\epsilon}^{\dot{\alpha}}\big(\tfrac{1}{2}\{D_\alpha, \bar{D}_{\dot{\alpha}}\} - \tfrac{1}{2}[D_\alpha, \bar{D}_{\dot{\alpha}}]\big)\mathbb{F}\big|,$$

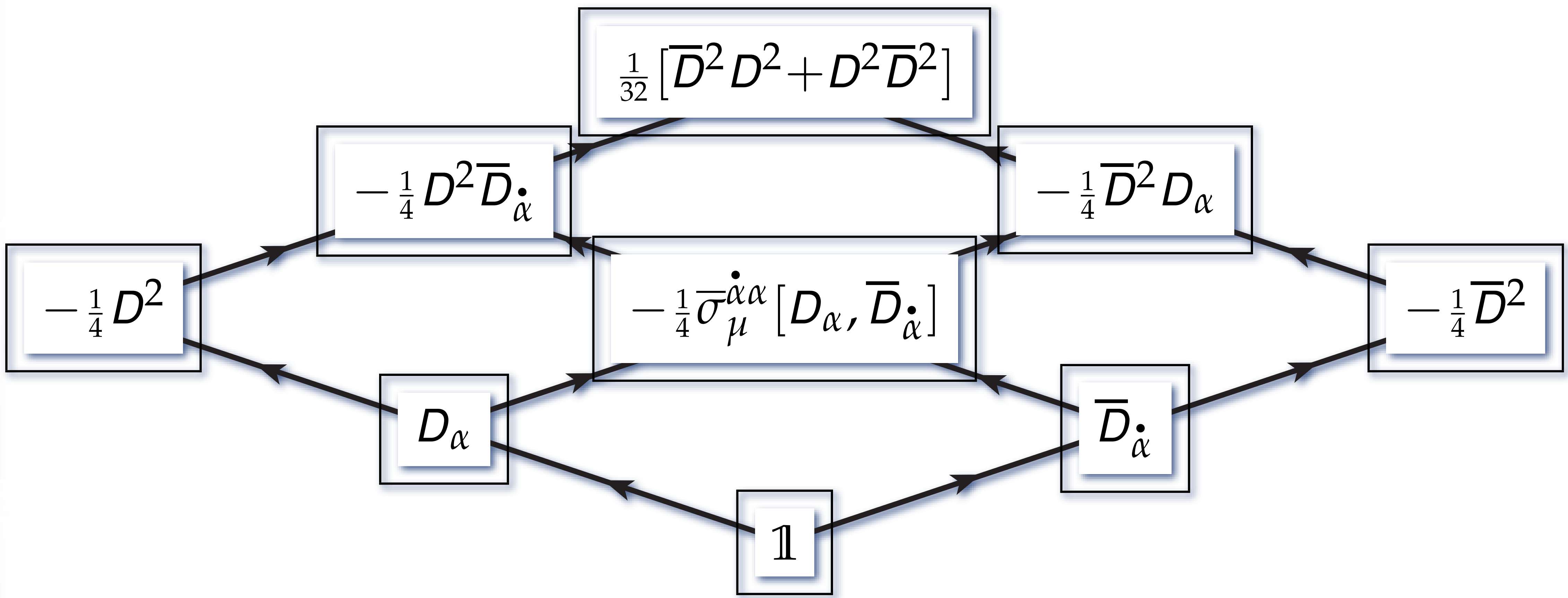
$$= \tfrac{1}{2}\epsilon^\beta\varepsilon_{\beta\alpha}(-4F) + i\hbar\sigma^\mu_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}(\partial_\mu\phi) - \tfrac{1}{4}\sigma^\mu_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}(\bar{\sigma}_\mu^{\dot{\beta}\beta}[D_\beta, \bar{D}_{\dot{\beta}}]\mathbb{F})\big|,$$

$$= 2\varepsilon_{\alpha\beta}\epsilon^\beta F + i\hbar\sigma^\mu_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}(\partial_\mu\phi) + \sigma^\mu_{\alpha\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}}F_\mu; \qquad \textit{etc.}$$

Superfields in 3+1-Dimensional Spacetime

Superfields

- Note:



$$\delta_Q(\epsilon) \int d^4x \, [D^2 \overline{D}^2 f(\mathbb{F}_1, \mathbb{F}_2, \dots)]| = \int d^4x \, \partial_\mu \mathcal{K}^\mu = 0$$

is a supersymmetric Lagrangian

Constrained and Gauge Superfields

Superconstrained Superfields

- For most applications, the complete intact superfield is too much.
- Super-constraining:

$$\textbf{chiral}: \quad \overline{D}_{\dot\alpha}\,\Phi = 0 \qquad \text{and} \qquad \textbf{anti-chiral}: \quad D_\alpha\,\overline{\Phi} = 0.$$

- These contain fewer component fields:

$$\phi := [\Phi]\big|, \qquad \psi_\alpha := [D_\alpha\Phi], \qquad F := -\tfrac{1}{4}[D^2\Phi]$$

- Other components reduce:

$$[\overline{D}_{\dot\alpha}\Phi]\big| = 0, \qquad [\overline{D}^2\Phi]\big| = 0,$$

$$[\overline{D}_{\dot\alpha}D_\alpha\Phi]\big| = [(2\sigma^\mu_{\alpha\dot\alpha}P_\mu - D_\alpha\overline{D}_{\dot\alpha})\Phi]\big| = [(-2i\sigma^\mu_{\alpha\dot\alpha}\hbar\partial_\mu)\Phi]\big| = -2i\hbar\sigma^\mu_{\alpha\dot\alpha}(\partial_\mu\phi).$$

- ...because of which

$$\delta_Q(\epsilon)\phi = (\epsilon\cdot D + \bar\epsilon\cdot\overline{D})\Phi\big| = \epsilon\cdot\psi;$$

$$\delta_Q(\epsilon)\psi_\alpha = (\epsilon\cdot D + \bar\epsilon\cdot\overline{D})D_\alpha\Phi\big| = (\tfrac{1}{2}\epsilon^\beta\varepsilon_{\beta\alpha}D^2 + 2i\hbar\bar\epsilon^{\dot\alpha}\sigma^\mu_{\alpha\dot\alpha}\partial_\mu)\Phi\big| = 2\varepsilon_{\alpha\beta}\epsilon^\beta F + 2i\hbar\sigma^\mu_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha}(\partial_\mu\phi);$$

$$\delta_Q(\epsilon)F = (\epsilon\cdot D + \bar\epsilon\cdot\overline{D})(-\tfrac{1}{4}D^2\Phi)\big| = -\tfrac{1}{4}\bar\epsilon^{\dot\alpha}(4i\hbar\sigma^\mu_{\alpha\dot\alpha}\varepsilon^{\alpha\beta}\partial_\mu D_\beta)\Phi\big| = -i\hbar\sigma^\mu_{\alpha\dot\alpha}\bar\epsilon^{\dot\alpha}\varepsilon^{\alpha\beta}(\partial_\mu\psi_\beta).$$

Constrained and Gauge Superfields

The Chiral Ring and Exact Results

- The product of two chiral superfields is again chiral.

$$\overline{D}_{\dot{\alpha}}\Phi_1 = 0 = \overline{D}_{\dot{\alpha}}\Phi_2, \quad \Rightarrow \quad \overline{D}_{\dot{\alpha}}(\Phi_1\Phi_2) = 0,$$

- with the usual distribution of multiplication over addition.
- Chiral superfields form a ring.
- In fact any analytic function of chiral superfields is also chiral.

$$\mathscr{L}[\Phi] = [D^2\overline{D}^2 K(\Phi^\dagger,\Phi)]| + [D^2 W(\Phi)]| + [\overline{D}^2\,\overline{W}(\Phi^\dagger)]|.$$

$$= -(\partial_\mu\phi^*)\eta^{\mu\nu}(\partial_\nu\phi) - \tfrac{i}{2}\overline{\sigma}^{\mu\dot{\alpha}\alpha}\left[\overline{\psi}_{\dot{\alpha}}(\partial_\mu\psi_\alpha) - (\partial_\mu\overline{\psi}_{\dot{\alpha}})\psi_\alpha\right] + F^*F$$

$$+ F\,W'(\phi) + \tfrac{1}{2}\varepsilon^{\alpha\beta}\psi_\alpha\psi_\beta\,W''(\phi) + F^*\,\overline{W}'(\phi^*) + \tfrac{1}{2}\varepsilon^{\dot{\alpha}\dot{\beta}}\overline{\psi}_{\dot{\alpha}}\overline{\psi}_{\dot{\beta}}\,\overline{W}''(\phi^*).$$

- The Euler-Lagrange equations imply

$$F^* = -W'(\phi) \qquad \text{and} \qquad F = -\overline{W}'(\phi^*)$$

$$\mathscr{L}[\Phi] = -\tfrac{i}{2}\overline{\sigma}^{\mu\dot{\alpha}\alpha}\left[\overline{\psi}_{\dot{\alpha}}(\partial_\mu\psi_\alpha) - (\partial_\mu\overline{\psi}_{\dot{\alpha}})\psi_\alpha\right] - (\partial_\mu\phi^*)\eta^{\mu\nu}(\partial_\nu\phi)$$

$$- \boxed{|W'(\phi)|^2} + \boxed{\tfrac{1}{2}\varepsilon^{\alpha\beta}\psi_\alpha\psi_\beta\,W''(\phi) + \tfrac{1}{2}\varepsilon^{\dot{\alpha}\dot{\beta}}\overline{\psi}_{\dot{\alpha}}\overline{\psi}_{\dot{\beta}}\,\overline{W}''(\phi^*)}.$$

non-negative potential

Constrained and Gauge Superfields

Supersymmetric Electrodynamics

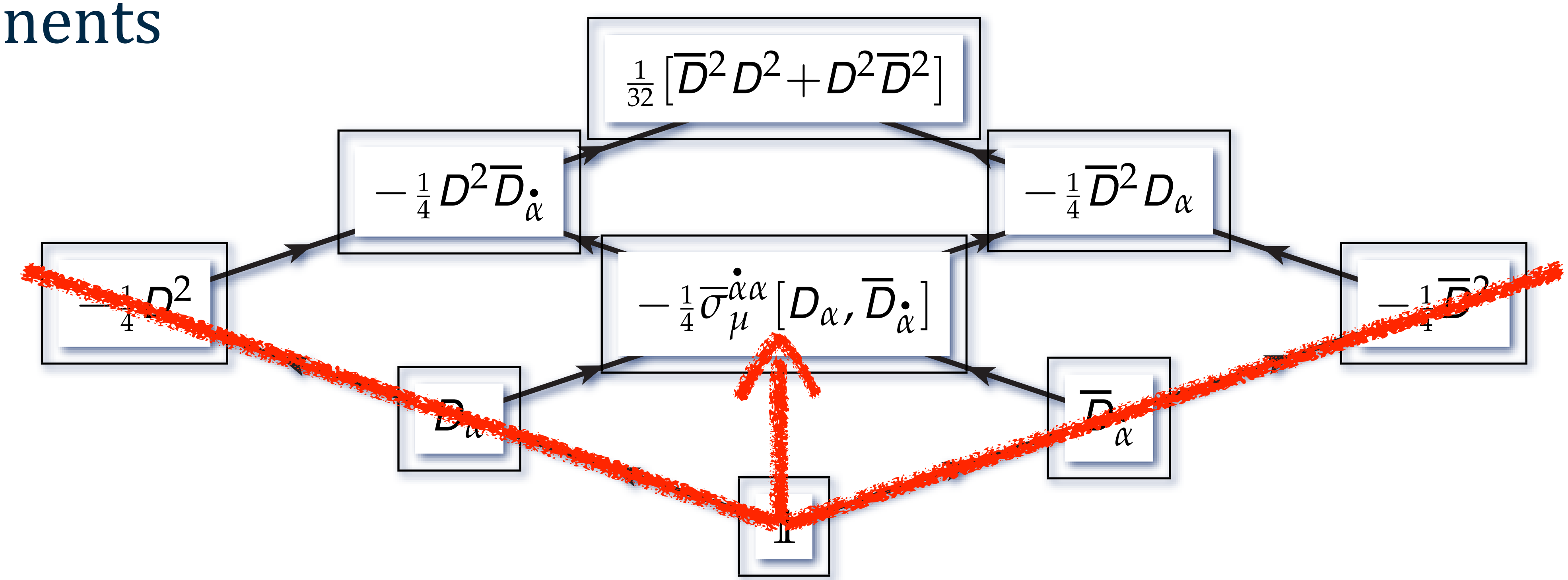
- Compute

$$\tfrac{1}{4}\bar{\sigma}_\mu^{\dot{\alpha}\alpha}\left[[D_\alpha,\bar{D}_{\dot{\alpha}}]i(\Phi-\Phi^\dagger)\right]\Big| = 2\hbar\left(\partial_\mu\,\Re e(\phi)\right)$$

- so

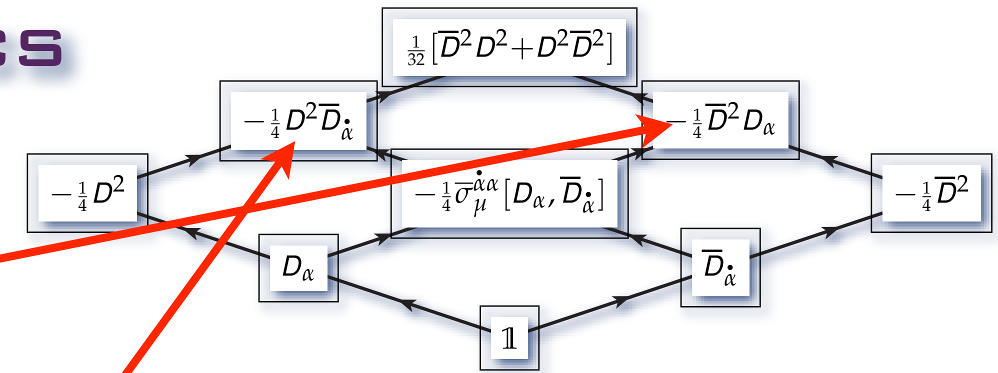
$$\mathbb{A} \;\to\; \mathbb{A}' = \mathbb{A} + i(\Phi-\Phi^\dagger) \qquad A_\mu \;\to\; A'_\mu = A_\mu + \partial_\mu\left(2\hbar\,\Re e(\phi)\right)$$

- Typically, use Φ for matter, and **Λ** for the gauge parameter.
- Then a judicious choice of **Λ** eliminates the lower half of the components



Constrained and Gauge Superfields

Supersymmetric Electrodynamics



- In the so gauge-fixed version of $\mathbb{A}^\dagger = \mathbb{A}$,
- project on the upper components:

$$\mathbb{A}_\alpha := (\overline{D}^2 D_\alpha \mathbb{A}) \qquad \text{and} \qquad \overline{\mathbb{A}}_{\dot\alpha} := (D^2 \overline{D}_{\dot\alpha} \mathbb{A}),$$

$$\mathbb{A} = \mathbb{A}^\dagger \quad \Rightarrow \quad \varepsilon^{\alpha\beta} D_\alpha \mathbb{A}_\beta = \varepsilon^{\dot\alpha\dot\beta} \overline{D}_{\dot\alpha} \overline{\mathbb{A}}_{\dot\beta},$$

$$\mathbb{A}_\alpha\big| =: \lambda_\alpha, \qquad \overline{\mathbb{A}}_{\dot\alpha}\big| =: \bar\lambda_{\dot\alpha},$$

$$D_\alpha \mathbb{A}_\beta\big| =: \varepsilon_{\alpha\beta} \mathrm{D} + i(\boldsymbol{\sigma}^{\mu\nu})_\alpha{}^\gamma \varepsilon_{\beta\gamma} F_{\mu\nu}, \qquad \overline{D}_{\dot\alpha} \overline{\mathbb{A}}_{\dot\beta}\big| =: \varepsilon_{\dot\alpha\dot\beta} \mathrm{D} + i(\overline{\boldsymbol{\sigma}}^{\mu\nu})^{\dot\gamma}{}_{\dot\alpha} \varepsilon_{\dot\beta\dot\gamma} F_{\mu\nu},$$

$$D^2 \mathbb{A}_\alpha\big| = -i\hbar \sigma^\mu_{\alpha\dot\alpha} \varepsilon^{\dot\alpha\dot\beta} (\partial_\mu \bar\lambda_{\dot\beta}), \qquad \overline{D}^2 \overline{\mathbb{A}}_{\dot\alpha}\big| = -i\hbar \sigma^\mu_{\alpha\dot\alpha} \varepsilon^{\alpha\beta} (\partial_\mu \lambda_\beta).$$

- so the supersymmetric generalization of the YM Lagrangian is

$$\mathscr{L}[\mathbb{A}] = -\tfrac{1}{4}[D^2 \varepsilon^{\alpha\beta} \mathbb{A}_\alpha \mathbb{A}_\beta]\big| - \tfrac{1}{4}[\overline{D}^2 \varepsilon^{\dot\alpha\dot\beta} \overline{\mathbb{A}}_{\dot\alpha} \overline{\mathbb{A}}_{\dot\beta}]\big|,$$

$$= -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{i\hbar}{2} \overline{\sigma}^{\mu\,\dot\alpha\alpha} \big[\bar\lambda_{\dot\alpha}(\partial_\mu \lambda_\alpha) - (\partial_\mu \bar\lambda_{\dot\alpha})\lambda_\alpha\big] + 2\mathrm{D}^2.$$

Constrained and Gauge Superfields

Supersymmetric Electrodynamics

- Finally, as for the interaction between matter and gauge fields,

$$\mathscr{L} = -\tfrac{1}{4}[D^2\, \varepsilon^{\alpha\beta} \mathbb{A}_\alpha \mathbb{A}_\beta]| - \tfrac{1}{4}[\overline{D}^2\, \varepsilon^{\dot\alpha\dot\beta} \overline{\mathbb{A}}_{\dot\alpha} \overline{\mathbb{A}}_{\dot\beta}]| + [D^2 \overline{D}^2\, \overline{\Phi}\, e^{q_\Phi \mathbb{A}}\, \Phi]|.$$

- This is invariant with respect to the gauge transformation

$$\mathbb{A} \to \mathbb{A} + i(\boldsymbol{\Lambda} - \overline{\boldsymbol{\Lambda}}), \qquad \Phi \to e^{iq_\Phi \boldsymbol{\Lambda}} \Phi, \qquad \overline{\Phi} \to e^{-iq_\Phi \overline{\boldsymbol{\Lambda}}} \overline{\Phi},$$

- and is manifestly supersymmetric.
- To include mass terms, need to introduce Φ^c:

$$\mathscr{L} = -\tfrac{1}{4}[D^2\, \varepsilon^{\alpha\beta} \mathbb{A}_\alpha \mathbb{A}_\beta]| - \tfrac{1}{4}[\overline{D}^2\, \varepsilon^{\dot\alpha\dot\beta} \overline{\mathbb{A}}_{\dot\alpha} \overline{\mathbb{A}}_{\dot\beta}]|$$
$$+ m\Big\{[D^2 \Phi \Phi^c]| + [\overline{D}^2 \overline{\Phi}\, \overline{\Phi^c}]|\Big\}$$
$$+ [D^2 \overline{D}^2\, \overline{\Phi}\, e^{q_\Phi \mathbb{A}}\, \Phi]| + [D^2 \overline{D}^2\, \overline{\Phi^c}\, e^{-q_\Phi \mathbb{A}}\, \Phi^c]|.$$

- Superpotential terms are not renormalized
- Gauge-interaction terms and kinetic terms for matter are

Thanks!

Tristan Hubsch

Department of Physics and Astronomy
Howard University, Washington DC
Prirodno-Matematički Fakultet
Univerzitet u Novom Sadu

http://homepage.mac.com/thubsch/