# (Fundamental) Physics of Elementary Particles

**Special solutions and singularities**
**Cosmological solutions and time-travel**
**Spacetime engineering and wormholes**

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# Fundamental Physics of Elementary Particles

## Program

- Special Solutions and Singularities
  - Massive, charged and rotating black holes
- Cosmological Solutions and Time-Travel
  - Standard geometries in cosmology
  - The cosmological constant & dark stuff
  - Non-standard cosmologies: Kasner, Gödel…
- Spacetime Engineering and Wormholes
  - Additivity of matter, but not of spacetime
  - Energy conditions
  - Einstein-Rosen bridge
    - What’s in the coordinate domain
    - Spacetime surgery
  - Traversable wormholes

# Special Solutions and Singularities

## Massive black holes

- 1915, Scwarzschild solution… a reminder

$$[g_{\mu\nu}] = \text{diag}\big(-f_S(r), \tfrac{1}{f_S(r)}, r^2, r^2\sin^2(\theta)\big),$$
$$\mathrm{d}s^2 = -f_S(r)c^2\mathrm{d}t^2 + \tfrac{1}{f_S(r)}\mathrm{d}r^2 + r^2\big(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2\big),$$
$$f_S(r) := \Big(1 - \frac{r_S}{r}\Big), \qquad r_S = \frac{2G_N\,M}{c^2}.$$

- Note that

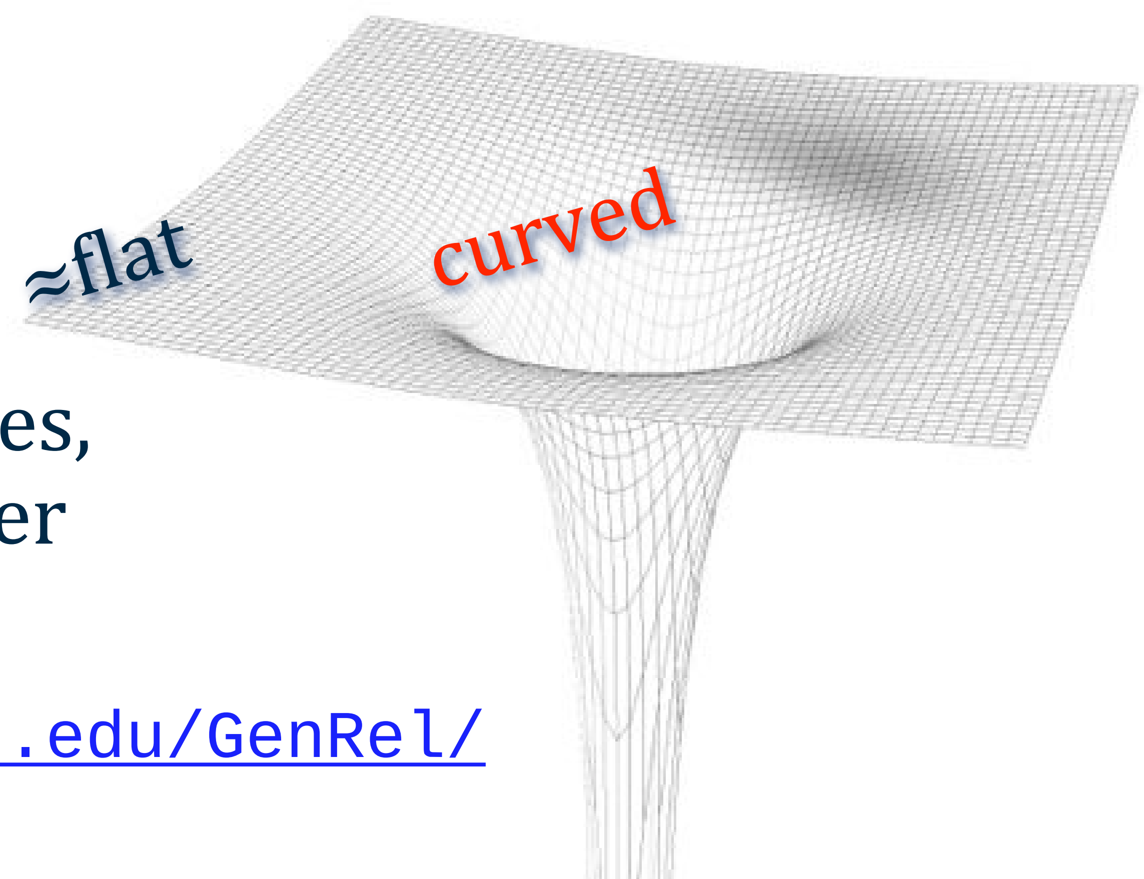
$$\Box f_S(r) = \frac{1}{r}\frac{\partial^2}{\partial r^2} r f_S(r) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r - r_S) \equiv 0$$

- That is, $f_S(r)$ is a harmonic function.
- This metric satisfies the Einstein equation w/o matter.
- The mass $M$ characterizes the spacetime itself
  - …and is localized at the origin, at the singularity,
  - …in that a Gaussian sphere can be “shrunk down” to it.

# Special Solutions and Singularities

## Massive black holes

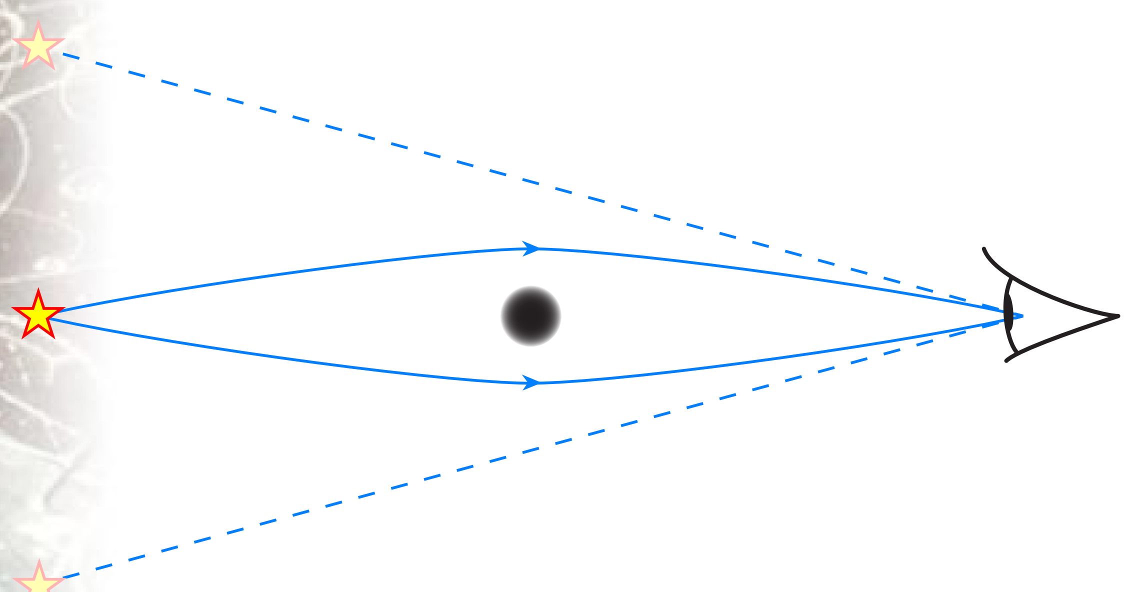
- Schwarzschild solution… a few more things (for now):
- It makes sense to put Gaussian encircling spheres only down to $r_S$, the event horizon: you cannot extract info from within.
- ...and, within $r_S$, a Gaussian sphere would have $t$ for the radius.
- For all practical purposes, the Schwarzschild geometry extends only down to $r_S$; the "inside" is inaccessible to external observers.
- The solution is asymptotically flat and spherically symmetric
- ...at $r \gg r_S$, approximately flat.
- There can easily be many black holes, sufficiently far away from each other no two to affect each other.
- Visit, e.g., [http://pisces.as.utexas.edu/GenRel/](http://pisces.as.utexas.edu/GenRel/)

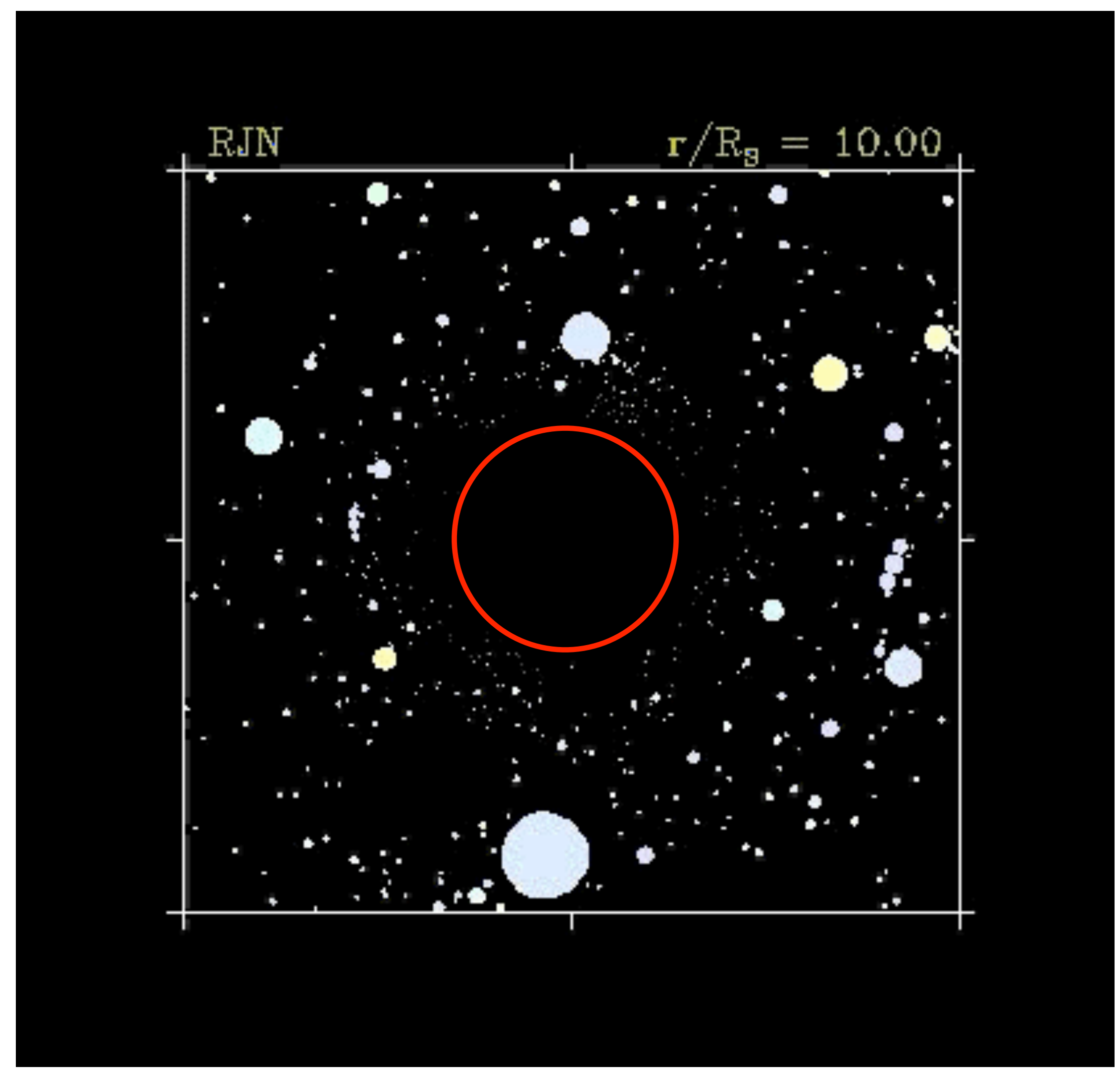


# Special Solutions and Singularities

## Massive black holes

- Schwarzschild solution… a few more things (for now):
- Any gravitational field bends geodesics—and so also light beams
- Light sources that pass behind a black hole have their light emitted to one and the other side bent around…
- ...producing a double image.





# Special Solutions and Singularities

## Massive and charged black holes

- 1916–1918, Hans Reissner and Gunnar Nordstrøm:

$$[g_{\mu\nu}] = \mathrm{diag}\big(-f_{RN}(r), \tfrac{1}{f_{RN}(r)}, r^2, r^2\sin^2(\theta)\big),$$

$$ds^2 = -f_{RN}(r)\,c^2 dt^2 + \tfrac{1}{f_{RN}(r)} dr^2 + r^2\big(d\theta^2 + \sin^2(\theta)\,d\varphi^2\big),$$

~~harmonic~~ $f_{RN}(r) := \Big(1 - \frac{r_S}{r} + \frac{r_q^2}{r^2}\Big), \qquad r_q := \sqrt{\frac{q^2\,G_N}{4\pi\epsilon_0\,c^4}}.$

- The characteristic function $f_{RN}(r)$ vanishes at:

$$r_\pm = \tfrac{1}{2}\Big(r_S \pm \sqrt{r_S^2 - 4r_q^2}\Big).$$

- There are two very different cases:
  - When $2r_q < r_S$: concentric horizons
  - When $2r_q > r_S$, *i.e.*, $\frac{q}{\sqrt{4\pi\epsilon_0}} > 4\sqrt{G_N}M$, the black hole is overcharged, there are no horizons, the singularity is accessible to all observers.

The singularity is "naked."

# Special Solutions and Singularities

## Massive and charged black holes

- And, there is the marginal case "in-between":
  - When $2r_q = r_S$: the horizons coincide    the extremal R-N solution
- ...which is a balancing act between the gravitational and the electrostatic field of the solution.
- Roger Penrose's "cosmic censorship hypothesis": that every singularity has an event horizon "screen."
- An overcharged Reissner-Nordstrøm black hole would thus violate Penrose's cosmic censorship hypothesis
- ...which is why it is ***believed*** that it is not possible to construct an overcharged black hole (a naked singularity) from scratch
- ...nonetheless, its mere existence is instructive
- A physically very nontrivial $g_{\mu\nu}$ satisfies the Einstein equations, without any matter added to support it.

# Special Solutions and Singularities

## Massive and rotating black holes

- 1963, Roy Kerr (1967, Robert H. Boyer & Richard H. Lindquist):

$$\mathrm{d}s^2 = -\Big(1-\frac{r_S\,r}{\rho^2}\Big)c^2\mathrm{d}t^2 + \rho^2\Big(\frac{1}{\Delta}\mathrm{d}r^2+\mathrm{d}\theta^2\Big)$$
$$+\Big(r^2+\ell^2+\frac{r_S\,r\,\ell^2}{\rho^2}\sin^2(\theta)\Big)\sin^2(\theta)\,\mathrm{d}\varphi^2 - \frac{2r_S\,r\,\ell\,\sin^2(\theta)}{\rho^2}\,c\,\mathrm{d}t\,\mathrm{d}\varphi,$$
$$\ell := \frac{L}{Mc},\qquad \rho := \sqrt{r^2+\ell^2\cos^2(\theta)},\qquad \Delta := r^2 - r_S\,r + \ell^2,$$

$\Delta = 0$: event horizon
Cauchy horizon
& a ring-like singularity within

- where $L$ is the angular momentum.
- The ($ct$, $r$, $\theta$, $\phi$) coordinates are not orthogonal.
- There are two pairs of "horizons":

$$r_{H\pm} = \tfrac{1}{2}\Big(r_S \pm \sqrt{r_S^2 - 4\ell^2}\Big) \qquad r_{E\pm} = \tfrac{1}{2}\Big[r_S \pm \sqrt{r_S^2 - 2\ell^2[1+\cos(\theta)]}\Big]$$

$g_{rr} \to \infty$ event horizon $\qquad g_{tt} \to 0$ ellipsoid "ergosphere"

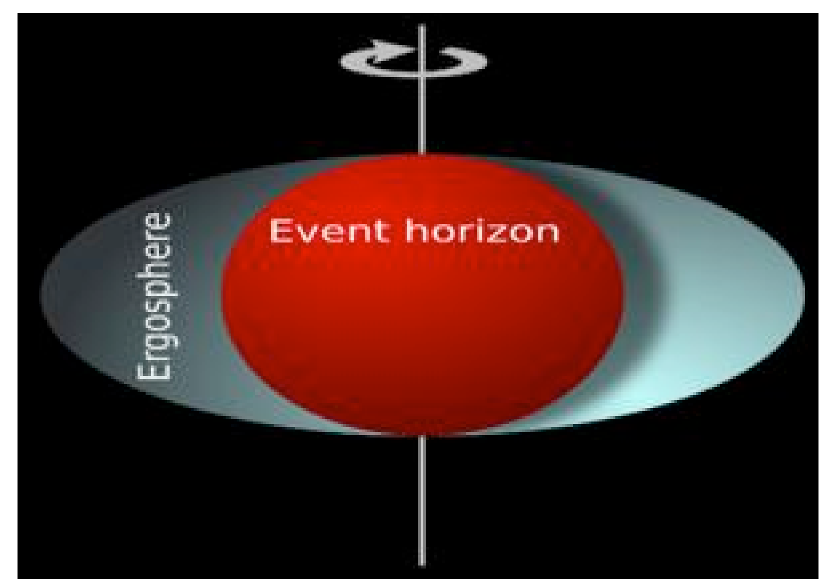
# Special Solutions and Singularities

## Massive and rotating black holes

- The region between the inner spherical event horizon and the outer ellipsoid is called the "ergosphere."
- Within the ergosphere, spacetime itself rotates with respect to the external observer, at the angular speed

$$\Omega = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = \frac{r_S\, r\, \ell\, c}{\rho^2(r^2+\ell^2) + r_S\, r\, \ell^2 \sin^2(\theta)}$$

- Objects that "dip" through the ergosphere must co-rotate
- ...even if this is faster than *c*, as seen from outside.
- Dipping into the ergosphere lets passing "parallel" outside objects
- ...extracts energy: Penrose's process.
- Possible to travel, through the ergosphere, back in time.



# Special Solutions and Singularities

## Massive, charged and rotating black holes

- 1965, Ezra Newman modified Kerr's solution:

$$\mathrm{d}s^2 = -\frac{\Delta}{\rho^2}\Big(c\,\mathrm{d}t - \ell\sin^2(\theta)\,\mathrm{d}\varphi\Big)^2 + \rho^2\Big(\frac{1}{\Delta}\mathrm{d}r^2 + \mathrm{d}\theta^2\Big) + \frac{\sin^2(\theta)}{\rho^2}\Big((r^2+\ell^2)\mathrm{d}\varphi - \ell c\,\mathrm{d}t\Big)^2,$$

$$\ell := \frac{L}{Mc}, \quad \rho := \sqrt{r^2+\ell^2\cos^2(\theta)}, \quad \Delta := r^2 - r_S\,r + \ell^2 + r_q^2, \quad r_q := \sqrt{\frac{q^2\,G_N}{4\pi\epsilon_0\,c^4}}$$

- The $(ct, r, \theta, \phi)$ coordinates are not orthogonal.
- The "horizon geometry" is considerably more complicated.
- Balancing acts between mass, angular momentum, and charge.
  - If $r_S^2 > 4(\ell^2+r_q^2)$, event horizon & ergosphere
  - If $r_S^2 < 4(\ell^2+r_q^2)$, no event horizon, no ergosphere
- Over-charging/spinning ⇒ no event horizon, naked singularity

# Special Solutions and Singularities

## Massive, charged and rotating black holes

- 1972–1973, Akira Tomimatsu & Humitaka Sato:

$$\mathrm{d}s^2 = -F\left[c\,\mathrm{d}t - G\,\mathrm{d}\varphi\right]^2 + F^{-1}\left[E\left(\mathrm{d}\rho^2 + \mathrm{d}z^2\right) + \rho^2 \mathrm{d}\varphi^2\right],$$

- in standard polar coordinates. The functions $E$, $F$, $G$ are however specified easiest using “prolate spheroidal” coordinates

$$x = \rho_0\sqrt{(\xi^2-1)(1-\eta^2)}\cos\varphi, \quad y = \rho_0\sqrt{(\xi^2-1)(1-\eta^2)}\sin\varphi,$$

$$z = \rho_0\,\xi\eta, \qquad \rho = \rho_0\sqrt{(\xi^2-1)(1-\eta^2)}$$

- Then:

$$E(\xi,\eta) := \frac{A(\xi,\eta)}{p^{2\delta}(\xi^2-\eta^2)^{\delta^2}}, \quad F(\xi,\eta) := \frac{A(\xi,\eta)}{B(\xi,\eta)}, \quad \rho_0 := \frac{G_N}{c^2}\frac{p}{\delta}m$$

$$G(\xi,\eta) := \frac{2L/mc}{A(\xi,\eta)}(1-\eta^2)C(\xi,\eta), \qquad p = \sqrt{1 - \frac{c^2}{{G_N}^2}\frac{L^2}{m^4}}$$

- where $A(\xi,\eta)$, $B(\xi,\eta)$, $C(\xi,\eta)$ are polynomials of degree $2\delta^2$, $2\delta^2$ and $(2\delta^2-1)$, respectively.

$\delta \in \mathbb{Z}$

# Special Solutions and Singularities

## Massive, charged and rotating black holes

- For $\delta = 1$, the Tomimatsu-Sato solution is equivalent to Kerr's
- For $\delta \neq 1$, the Tomimatsu-Sato solutions have naked singularities
- These solutions are but a select few of a large class of known, exact solutions to the Einstein equations
- ...many of which with various spacetime singularities
- ...and mass, charge and angular momentum values.
- So, could the electron be (modeled as) a charged black hole?
- With standard $q_e = 1.602\,176{\times}10^{-19}$ C & $m_e = 9.109\,382{\times}10^{-31}$ kg,

$$r_q(e^-) = 9.152 \times 10^{-37}\,\text{m} < \ell_P$$

$$r_S(e^-) = 1.353 \times 10^{-57}\,\text{m} \ll \ell_P$$

- This model is not wrong, but quite ***pointless***: there would exist no directly observable consequence. There are indirect problems...

# Cosmological Solutions and Time-Travel

## Standard geometries in cosmology

- Alexander Friedman, Georges H.J.E. Lemaître, Howard P. Robertson and Arthur G. Walker (FLRW):

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\mathrm{d}\Sigma^2, \qquad \begin{cases} \mathrm{d}\Sigma^2 := \left[\dfrac{\mathrm{d}r^2}{1-Kr^2} + r^2\mathrm{d}\Omega^2\right], \\ \mathrm{d}\Omega^2 := \mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\varphi^2 \end{cases}$$

- ...$a(t)$ is the scale function, $K$ the Gauss curvature at $a(t) = 1$.
- These $(t, r, \theta, \varphi)$ coordinates cover only half the spacetime
- Or use "hyper-spherical coordinates":

$$\mathrm{d}\Sigma^2 = \mathrm{d}r^2 + S_K{}^2(r)\mathrm{d}\Omega^2, \quad S_K(r) := \begin{cases} \frac{1}{\sqrt{K}}\sin(r\sqrt{K}) & K > 0, \\ r & K = 0, \\ \frac{1}{\sqrt{|K|}}\sinh(r\sqrt{|K|}) & K < 0. \end{cases}$$

# Cosmological Solutions and Time-Travel

## The cosmological constant & dark stuff

- The “standard form” of Einstein’s equations

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \frac{8\pi\, G_N}{c^4} T_{\mu\nu},$$

- is not what he originally published. Instead,

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \frac{8\pi\, G_N}{c^4} T_{\mu\nu} - g_{\mu\nu}\Lambda$$

- ...where $\Lambda$ is the cosmological constant.
- Motivated by:
  - the possibility of adding to the Einstein-Hilbert action $\int \sqrt{-g}\,\mathrm{d}^4x\,\Lambda$
  - the fact that $\Lambda$ permits a stationary flat geometry

$$\left(\frac{\dot a}{a}\right)^2 + \frac{Kc^2}{a^2} - \cancel{\frac{\Lambda c^2}{3}} = \frac{8\pi G_N}{3}\,\varrho, \qquad \varrho \to \varrho - \cancel{\frac{\Lambda c^2}{8\pi G_N}}$$

$$2\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 + \frac{Kc^2}{a^2} - \cancel{\Lambda c^2} = -\frac{8\pi G_N}{c^2}\,p, \qquad p \to p + \cancel{\frac{\Lambda c^4}{8\pi G_N}}$$

# Cosmological Solutions and Time-Travel

## The cosmological constant & dark stuff

- Thus, any (matter distribution with $p = -\rho c^2$) = $\Lambda$.
- In general (for isotropic & homogeneous matter)
  - **Dark energy**: anything that has $p/\rho < 0$.
  - **Quintessence**: anything that has $p/\rho < -c^2/3$.
  - **Cosmological constant**: anything that has $p/\rho = -c^2$.
  - **Phantom energy**: anything that has $p/\rho < -c^2$.
- Of particular interest:

You are here.

$$ds^2 = \begin{cases} -c^2 \mathrm{d}t^2 + a_0^2\, e^{+2c\sqrt{\Lambda/3}\,t}\, \mathrm{d}\vec{r}^{\,2}, & \textbf{de Sitter}, \\ -c^2 \mathrm{d}t^2 + \mathrm{d}\vec{r}^{\,2}, & \textbf{Minkowski}, \\ -c^2 \mathrm{d}t^2 + a_0^2\, e^{-2c\sqrt{\Lambda/3}\,t}\, \mathrm{d}\vec{r}^{\,2}, & \textbf{anti de Sitter}, \end{cases}$$

$$ds^2 = -c^2\left(1 \mp \tfrac{1}{3}\Lambda\rho^2\right)\mathrm{d}\tau^2 + \left(1 \mp \tfrac{1}{3}\Lambda\rho^2\right)^{-1}\mathrm{d}\rho^2 + \rho^2\left(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2\right)$$

$H := 2\sqrt{\Lambda/3} > 0$ is the Hubble constant.

# Cosmological Solutions and Time-Travel

## Nonstandard geometries in cosmology

- 1921, Edward Kasner (w/o matter support):

$$ds^2 = -c^2 dt^2 + \sum_{i=1}^{3} \left(\tfrac{t}{T_i}\right)^{2p_i} (dx^i)^2,$$

- where

$$\textstyle\sum_{i=1}^{3} p_i \;=\; 1 \;=\; \sum_{i=1}^{3} (p_i)^2.$$

- If any two of $p_i$ are set to vanish, the whole Riemann tensor vanishes — yet, the spacetime is neither flat nor isotropic.

$$p_2^{\pm} = \tfrac{1}{2}\left(1 - p_1 \pm \sqrt{1 + 2p_1 - 3{p_1}^2}\right),$$

$$p_3^{\pm} = 1 - p_1 - \tfrac{1}{2}\left(1 - p_1 \pm \sqrt{1 + 2p_1 - 3{p_1}^2}\right),$$

  - so $-1/3 \leq p_i \leq 1$: permutations of (0,0,1) ... (–1/3,2/3,2/3).
  - Spacetime volume expands linearly in coordinate time:

$$\sqrt{-g} = ct/({T_1}^{p_1}{T_2}^{p_2}{T_3}^{p_3})$$

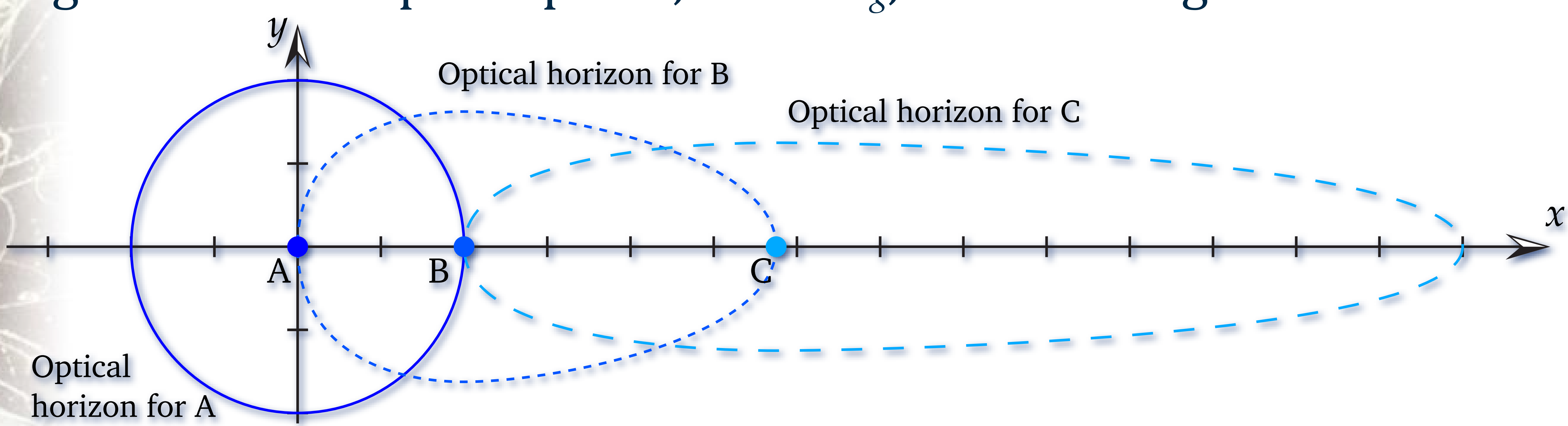
# Cosmological Solutions and Time-Travel

## Nonstandard geometries in cosmology

- 1949, Kurt Gödel:

$$ds^2 = -c^2 dt^2 + \frac{dr^2}{1+\left(\frac{r}{r_g}\right)^2} + r^2\left[1-\left(\frac{r}{r_g}\right)^2\right]d\phi^2 + dz^2 - c\,\frac{2\sqrt{2}\,r^2}{r_g}\,dt\,d\phi,$$

- The cylindrical coordinates $(t, r, \phi, z)$ co-rotate. $\Omega_g := \frac{\sqrt{2}\,c}{r_g}$
- Light follows elliptical paths, out to $r_g$, then turning back.




- Massive particles at rest continue moving only in time.

# Cosmological Solutions and Time-Travel

## Nonstandard geometries in cosmology

- Isometries of Gödel's metric:

$$X_0 := \frac{1}{\Omega_g}\partial_t, \qquad X_3 := \partial_z, \qquad \text{and} \qquad X_2 := \partial_\phi.$$

- Far less obviously:

$$X_{1,4} := \frac{1}{\sqrt{1+\left(\frac{r}{r_g}\right)^2}}\Big[\frac{r}{\sqrt{2}c}\cos\phi\,\partial_t \pm \frac{r_g}{2}\Big[1+\left(\frac{r}{r_g}\right)^2\Big]\Big\{{\sin\phi \atop \cos\phi}\Big\}\partial_r + \frac{r_g}{2r}\Big[1+2\left(\frac{r}{r_g}\right)^2\Big]\Big\{{\cos\phi \atop \sin\phi}\Big\}\partial_z\Big]$$

- and

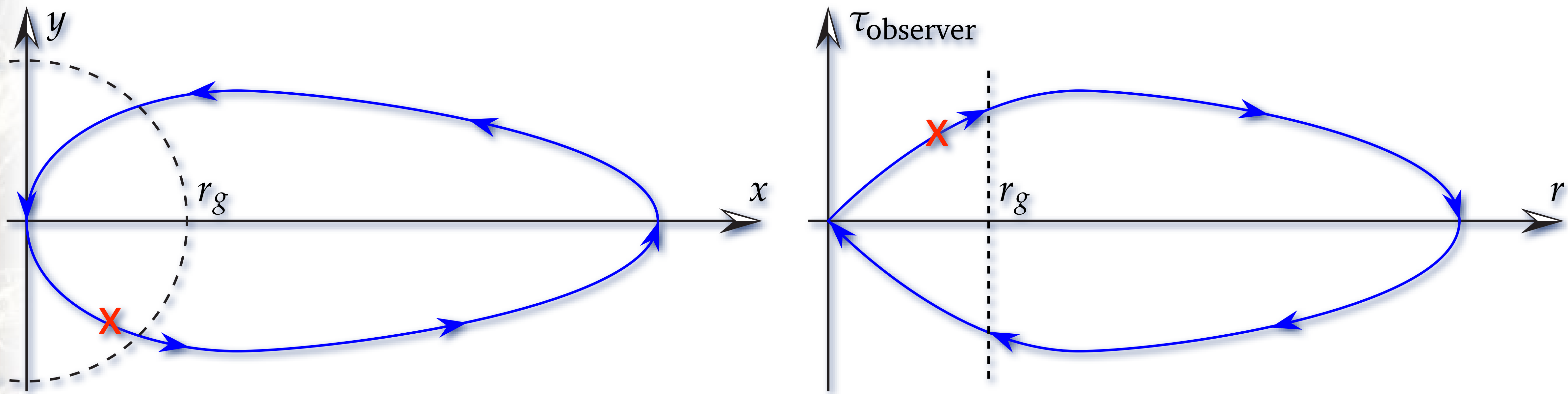
$$L_1 := X_4,\ \ L_2 := X_1, \quad L_3 := -i(X_0+X_2), \qquad \begin{cases} [\,L_j\,,\,L_k\,] = i\varepsilon_{jk}{}^{\ell}L_\ell, \\ [\,L_j\,,\,X_0\,] = 0 = [\,L_j\,,\,X_3\,], \end{cases}$$

- generates the $\mathfrak{so}(3)\oplus\mathfrak{tr}(\mathbb{R}^{1,1})$ algebra.
  - Acts "***transitively***": find paths that include the origin
  - …then transform that point to any other.

# Cosmological Solutions and Time-Travel

## Nonstandard geometries in cosmology

- Gödel's universe is geodesically complete, yet has no singularity
- ...and has an unusually high degree of isometry: $\mathfrak{so}(3) \oplus \mathfrak{tr}(\mathbb{R}^{1,1})$.
- Traveling in time: "Look, ma: no singularity!"




- but, you must start:
  - from $r_{ini} < 1.7\, r_g$
  - and with $v_{ini} > 0.98c$

...or any other initial data obtained from this, using the $\mathfrak{so}(3) \oplus \mathfrak{tr}(\mathbb{R}^{1,1})$ isometry.

# Cosmological Solutions and Time-Travel

## Nonstandard geometries in cosmology

- Einstein tensor:

$$[R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R] = T_{\mu\nu}$$
$$= \Omega_g^2 \,\mathrm{diag}(-1,1,1,1) + 2\Omega_g^2 \,\mathrm{diag}(1,0,0,0)$$

  - The 1st part: “lambda vacuum” = sol’n w/$\Lambda$.
  - The 2nd part: co-rotating perfect fluid/dust.
- Notice: energy momentum tensors are additive
- ...provided the matter distributions can co-exist

**Einstein tensors and energy-momentum density tensors of matter/energy distributions are additive; the corresponding metrics are not.**

# Spacetime Engineering and Wormholes

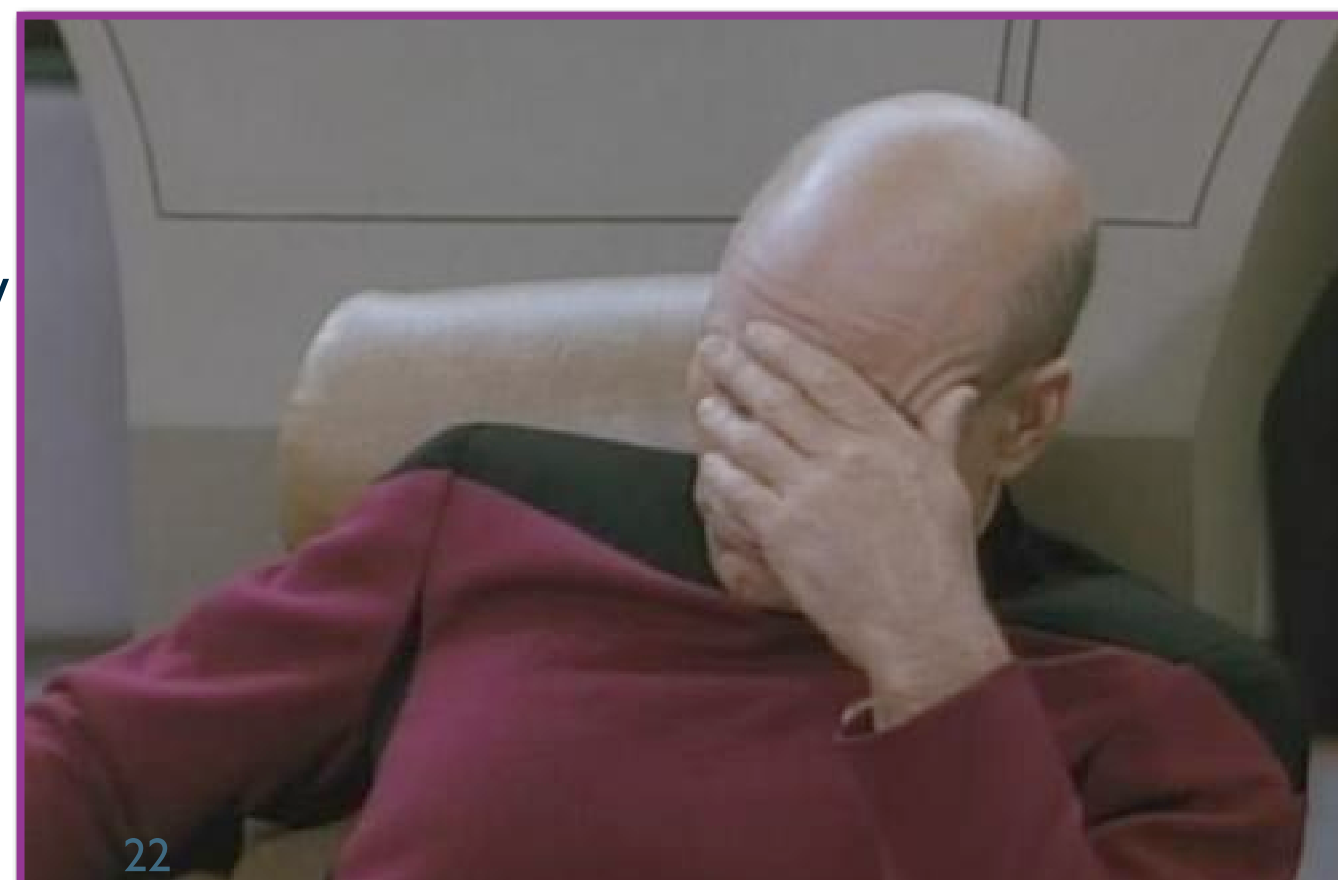
## Time travel

- In short: time travel is perfectly possible in general relativity.
- Closed time-like curves (CTC)
  - In Gödel's universe, the high degree of symmetry makes it possible to prove that there can be no causality violation.
  - Traveling through the ergosphere of the Kerr geometry, or its ring-like singularity, or many other constructions…
  - …semi-classical arguments: causality violation is probably precluded.
- 1992, Stephen Hawking: "general chronology protection principle" (hypothesis)
- 1975, Igor Novikov: only self-consistent CTC's are possible.
- Chronology violating set (CVS?) = points traversed by CTCs
- The boundary of CVS is the Cauchy horizon, generated by closed null geodesics.

# Spacetime Engineering and Wormholes

## Energy conditions

- Use Einstein equations as you would the Gauss-Ampère ones:
- Specify a desired geometry, *i.e.*, $\{ (t, \xi, \eta, \zeta), g_{\mu\nu} \}$
- Compute the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - ½\, g_{\mu\nu} R$
- Identify the energy-momentum density tensor
- as a sum of matter/energy components.
  - Physical characteristics of matter/energy distributions?
  - Can such matter/energy be assembled from known types/forms of matter/energy
  - ...or does it require ***exotic*** matter/energy?
- Make it so!



# Spacetime Engineering and Wormholes

## Energy conditions

- To aid in this characterization, define:
  1. a **time-like** 4-vector field with components $\xi^\mu(\mathrm{x})$, *i.e.*, $g_{\mu\nu}\xi^\mu\xi^\nu < 0, \forall\mathrm{x}$;
  2. a **light-like**, *i.e.*, null-vector field w/components $k^\mu(\mathrm{x})$, *i.e.*, $g_{\mu\nu}k^\mu k^\nu = 0, \forall\mathrm{x}$;
  3. a **causal** 4-vector field with components $\zeta^\mu(\mathrm{x})$, *i.e.*, $g_{\mu\nu}\zeta^\mu\zeta^\nu \leqslant 0, \forall\mathrm{x}$.

| | Condition | for all |
|---|---|---|
| **Dominant** | $g^{\mu\nu}T_{\mu\rho}T_{\nu\sigma}\zeta^\rho\zeta^\sigma \leqslant 0$ and $g^{0\mu}T_{\mu\nu}\zeta^\nu < 0$ | $g_{\mu\nu}\zeta^\mu\zeta^\nu \leqslant 0, \ (\zeta^0 > 0)$ |
| **Weak** | $T_{\mu\nu}\xi^\mu\xi^\nu \leqslant 0$ | $g_{\mu\nu}\xi^\mu\xi^\nu < 0$ |
| **Null*** | $T_{\mu\nu}k^\mu k^\nu \leqslant 0$ | $g_{\mu\nu}k^\mu k^\nu = 0$ |
| **Strong** | $\left[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right]\xi^\mu\xi^\nu \leqslant 0$ | $g_{\mu\nu}\xi^\mu\xi^\nu < 0$ |

* The "Null" condition is also often referred to as "light-like."

**Dominant** $\Rightarrow$ **weak** $\Rightarrow$ **Null** $\Leftarrow$ **Strong**

# Spacetime Engineering and Wormholes

## Einstein-Rosen bridge

- Recall Scwarzschild's solution:

$$\mathrm{d}s^2 = -f_S(r)c^2\mathrm{d}t^2 + \frac{1}{f_S(r)}\mathrm{d}r^2 + r^2\big(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2\big),$$

$$f_S(r) := \Big(1 - \frac{r_S}{r}\Big), \qquad r_S = \frac{2G_N M}{c^2}.$$

- @ $r = r_S$,
  1. the time component, $g_{00} = g_{tt} = -\big(1-\frac{r_S}{r}\big)c^2$ vanishes,
  2. the radial component, $g_{rr} = -\big(1-\frac{r_S}{r}\big)^{-1}$ diverges.
- For $r < r_S$,

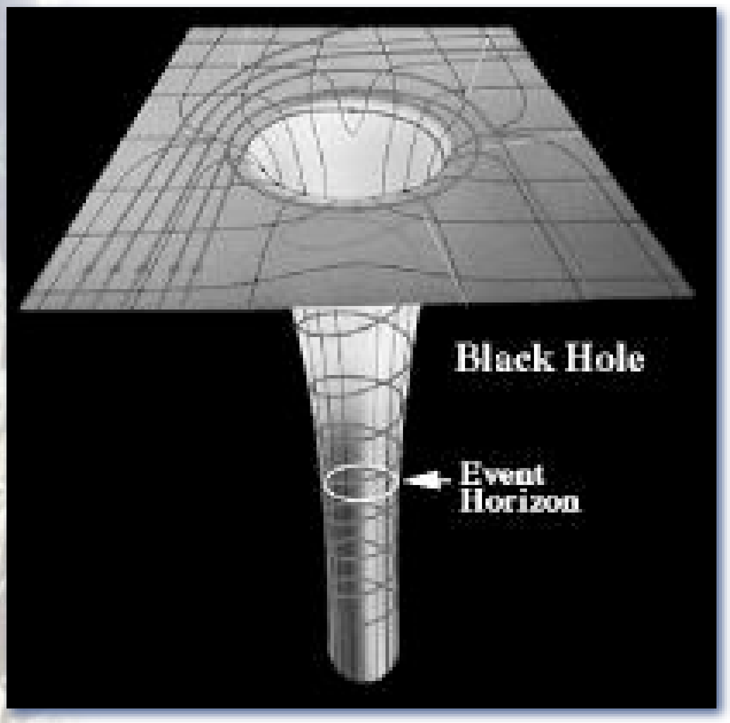
$$f_S(r) < 0 \quad \text{so} \quad g_{tt} = -f_S(r) > 0 \ \text{ and } \ g_{rr} = \big(f_S(r)\big)^{-1} < 0.$$

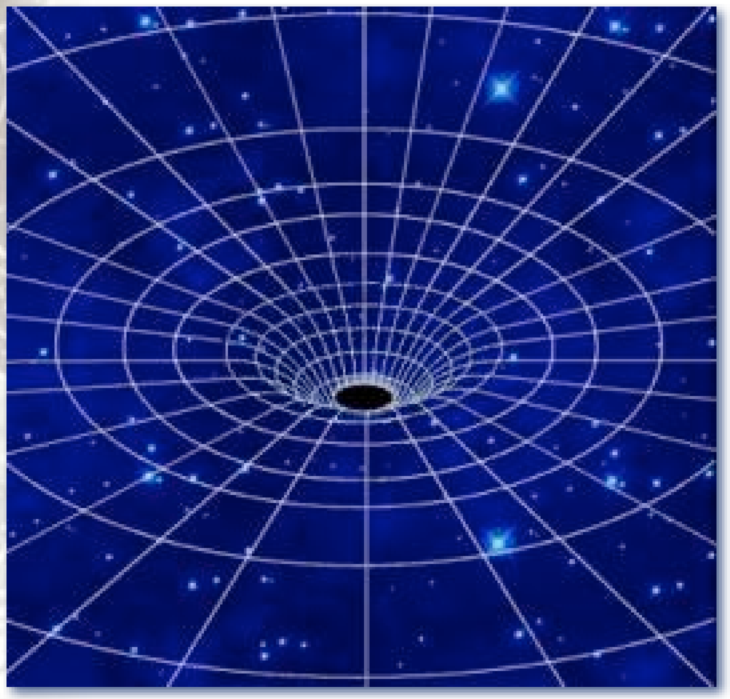
- As seen by an outside observer,
  - all speeds come to a screeching halt near the event horizon
  - in-falling objects take forever to reach the event horizon
  - out-coming light becomes red-shifted to $\lambda \to \infty$.

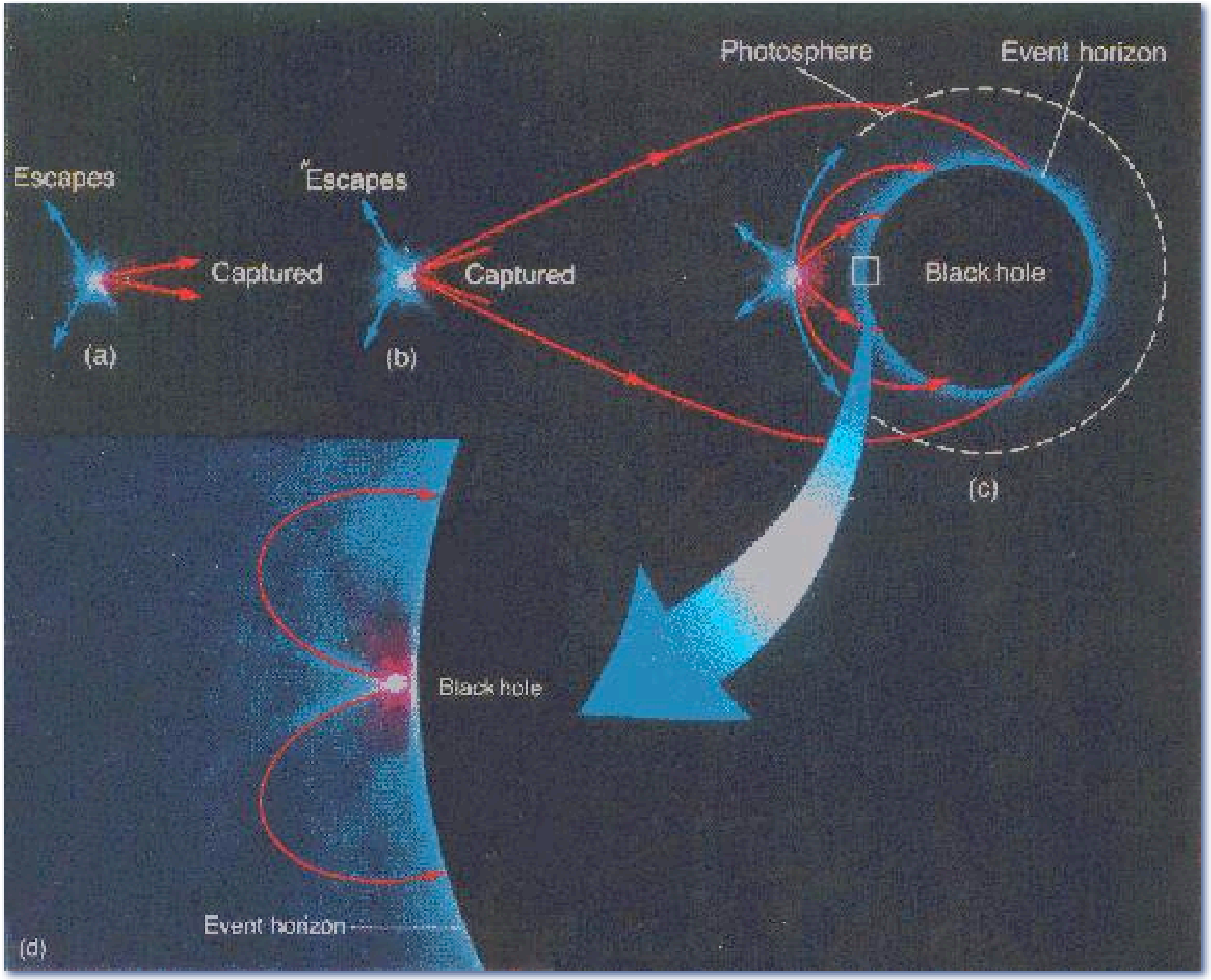
# Spacetime Engineering and Wormholes

## Einstein-Rosen bridge

- Some picturesque info on Scwarzschild's solution:






 http://cosmology.com/BlackHoles.html

# Spacetime Engineering and Wormholes

## Einstein-Rosen bridge

- Scwarzschild's solution, in Kruskal-Szekeres coordinates:

| K-Sz | | Schwarzschild |
|---|---|---|
| $u_I, \; -u_{III}$ | $=$ | $\sqrt{\frac{r}{r_S}-1}\, e^{r/r_S} \cosh\left(\frac{ct}{2r_S}\right)$ |
| $v_I, \; -v_{III}$ | $=$ | $\sqrt{\frac{r}{r_S}-1}\, e^{r/r_S} \sinh\left(\frac{ct}{2r_S}\right)$ |

1950: J.L. Synge discovered incompleteness & a system of complete coordinates.

1959: C. Fronsdal re-discovered incompleteness & a constrained description.

| K-Sz | | Schwarzschild |
|---|---|---|
| $u_{II}, \; -u_{IV}$ | $=$ | $\sqrt{1-\frac{r}{r_S}}\, e^{r/r_S} \sinh\left(\frac{ct}{2r_S}\right)$ |
| $v_{II}, \; -v_{IV}$ | $=$ | $\sqrt{1-\frac{r}{r_S}}\, e^{r/r_S} \cosh\left(\frac{ct}{2r_S}\right)$ |

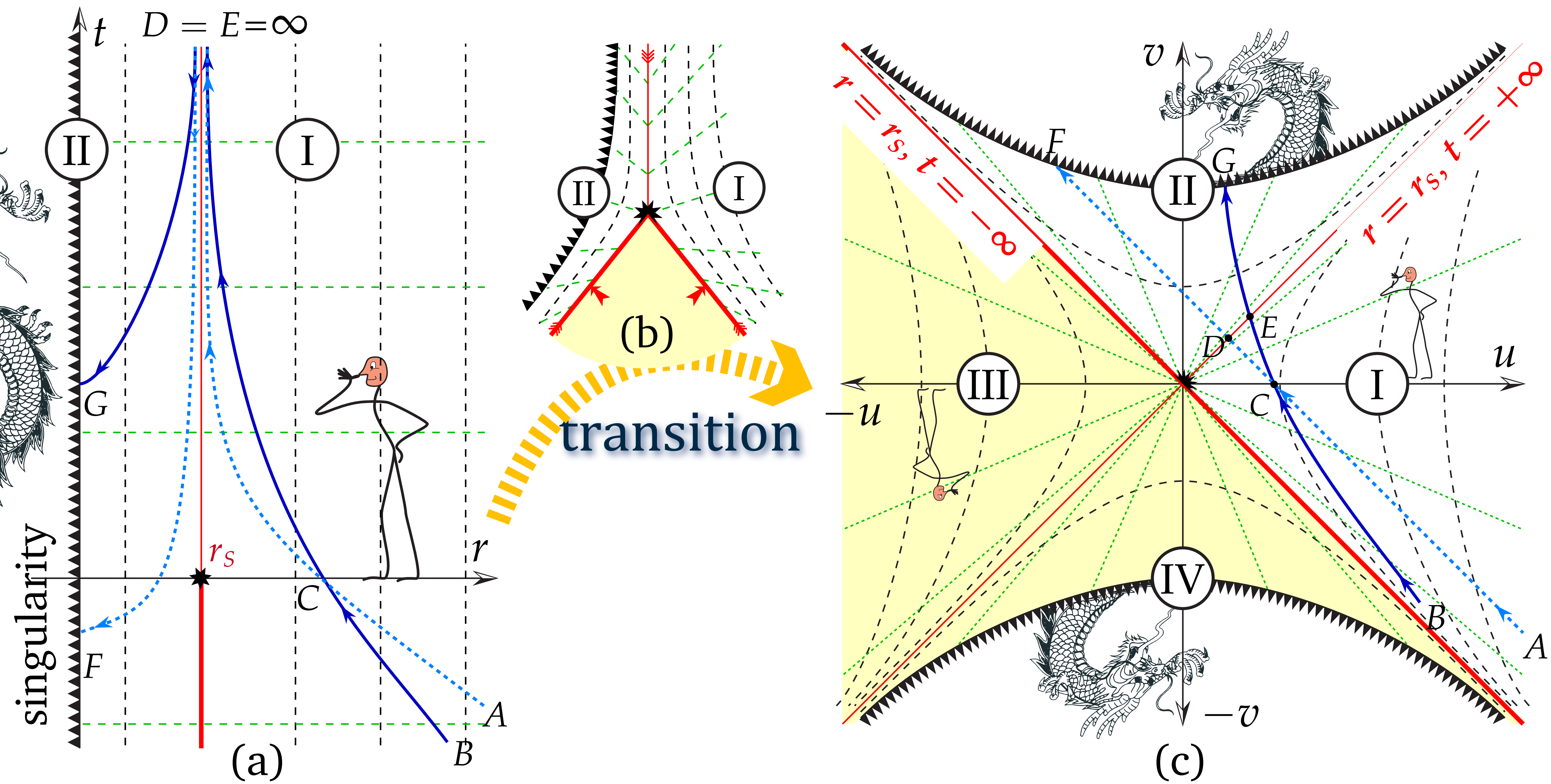
$$\left(\frac{r}{r_S}-1\right) e^{r/r_S} = u^2 - v^2,$$

$$t = \begin{cases} \frac{2r_S}{c} \operatorname{arth}\left(\frac{v}{u}\right) & \text{in regions I and III;} \\ \frac{2r_S}{c} \operatorname{arth}\left(\frac{u}{v}\right) & \text{in regions II and IV;} \end{cases}$$

# Spacetime Engineering and Wormholes

## Einstein-Rosen bridge

- Recall Schwarzschild's solution:



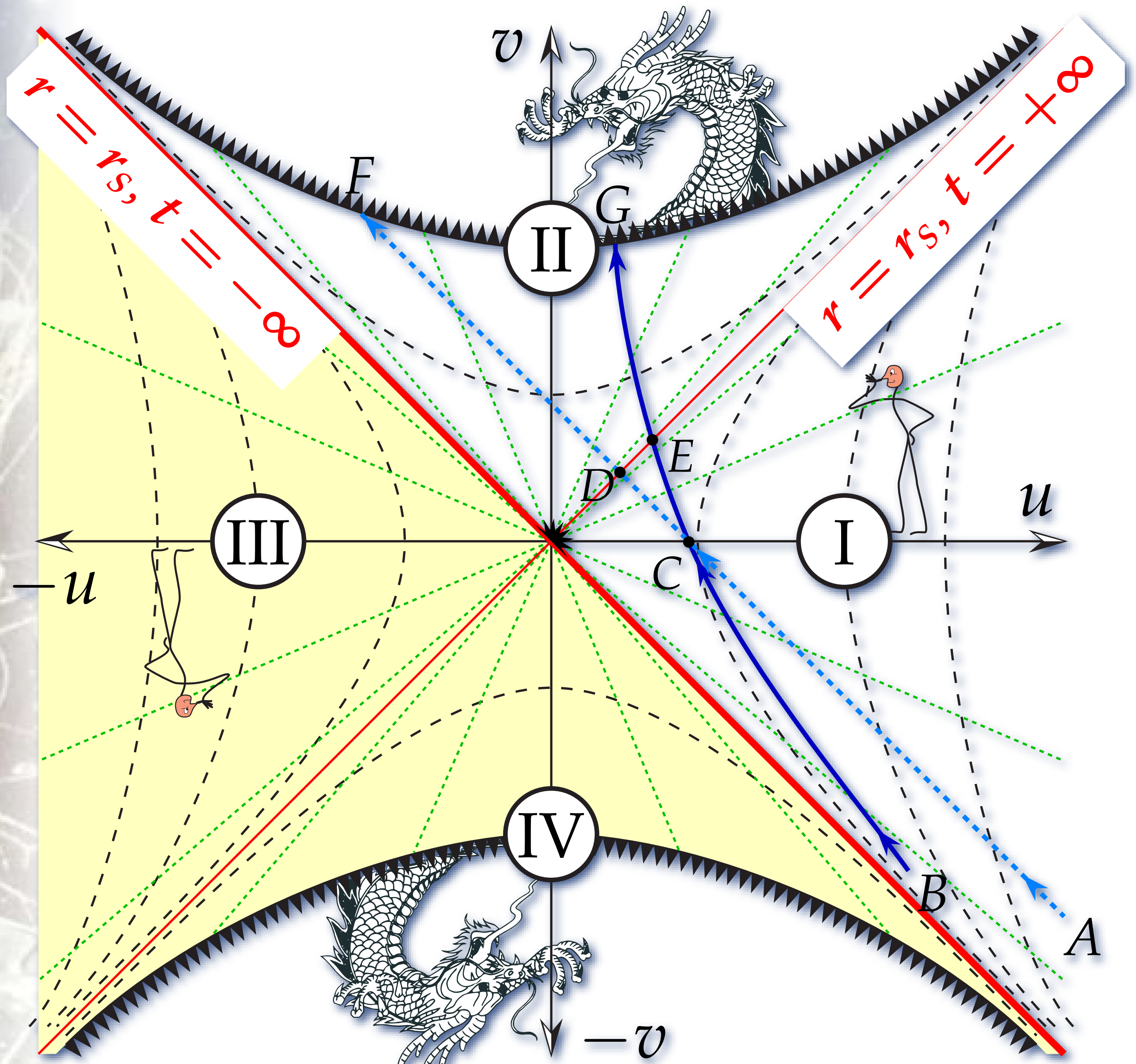

Schwarzschild coordinates

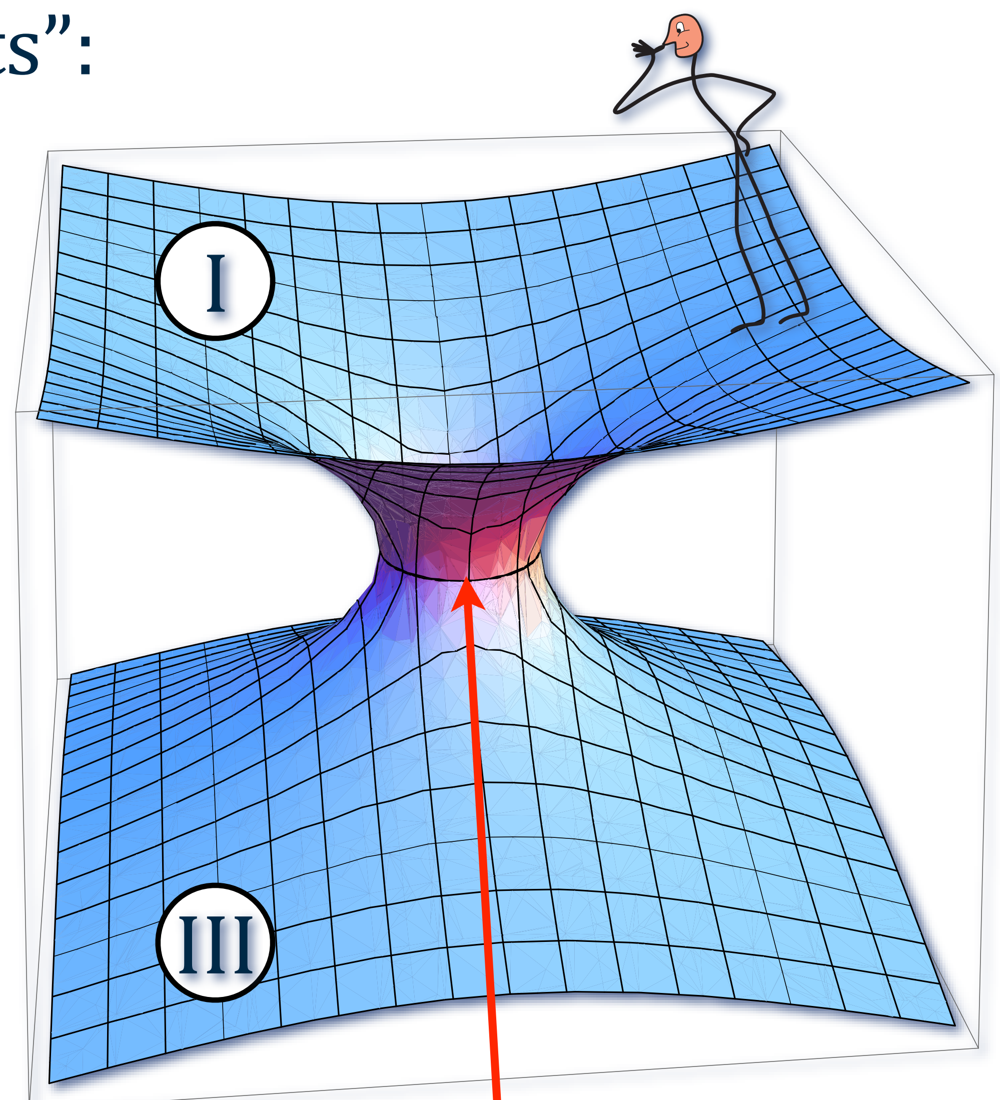
Kruskal-Szekeres coordinates

# Spacetime Engineering and Wormholes

## Einstein-Rosen bridge

- Scwarzschild's solution has two "sheets":



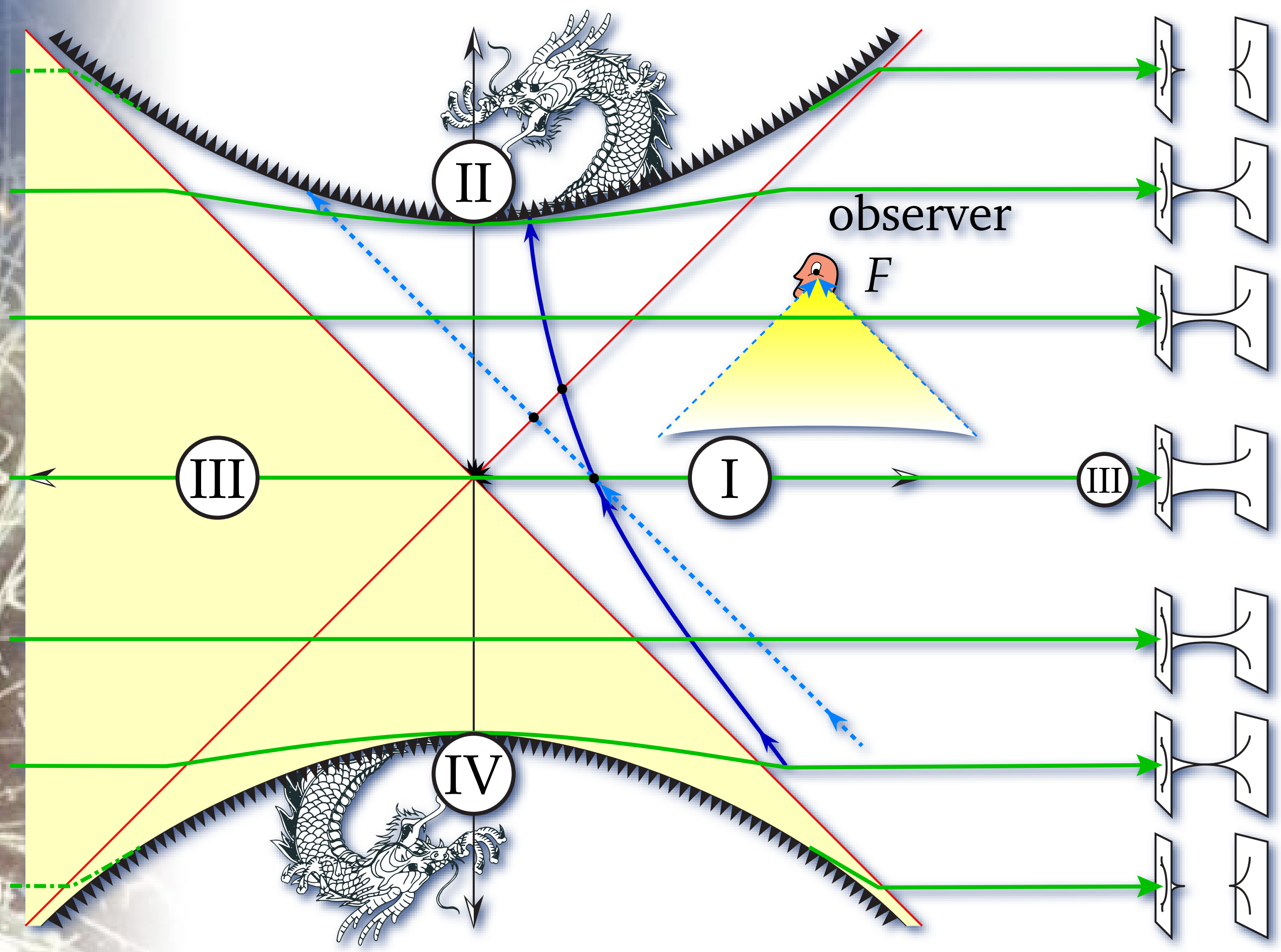




the seam forms the "throat"

# Spacetime Engineering and Wormholes

## Einstein-Rosen bridge

- Although static, Scwarzschild's solution has a dynamical "story":




Two separates "sides" of a black hole, one side seen from region I, the other from region III

The Einstein-Rosen bridge closes.

The Einstein-Rosen bridge begins to close; regions I and III are still spatially connected.

I The Einstein-Rosen bridge is maximally open.

The Einstein-Rosen bridge partially open; region I and III are spatially connected.

The Einstein-Rosen bridge is about to open.

Two separate "sides" of a black hole, one side seen from region I, the other from region III

The Einstein-Rosen bridge is closed for massive and even massless real particles.

But not for virtual particles.

# Spacetime Engineering and Wormholes

## Einstein-Rosen bridge

- The dynamical story of the Scwarzschild's solution, i.e., the Einstein-Rosen bridge connecting two regions of spacetime that:

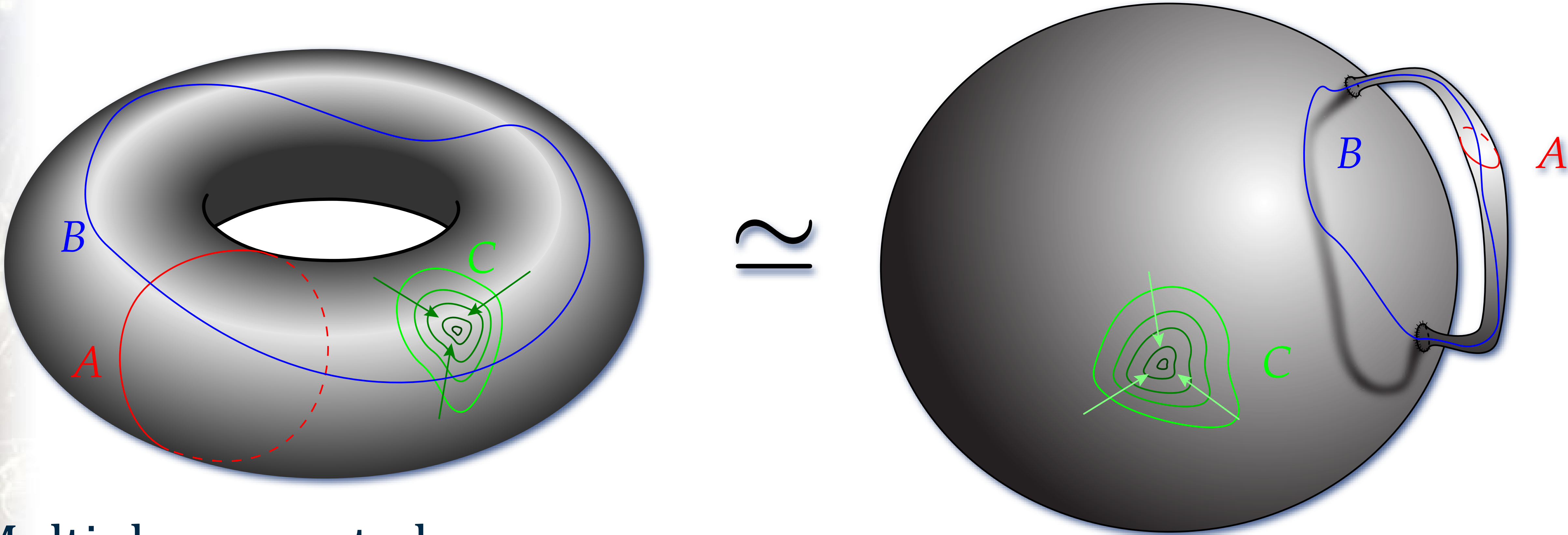
  1. have a black hole each;
  2. these two black holes connect in a moment;
  3. the connection of these black holes opens into a space-like "bridge" (wormhole) of the $S^2 \times \mathbb{R}^1$ topology;
  4. this "bridge" closes before even light could pass through it;
  5. there remain two separated regions, with a black hole each.

- Natural question: are there traversable "bridges"?
  - Called "Lorentzian wormholes"
  - Typically need matter for support.
  - Typically need exotic matter.

# Spacetime Engineering and Wormholes

## Traversable wormholes

- A little topological digression:




- Multiple connectedness:
  - The loops A and B cannot be shrunk continuously to a point.
  - Neither of them is the boundary of a region of space(time).
  - The loop C can be shrunk continuously to a point.
  - The loop C is a boundary of a region of space(time).

# Spacetime Engineering and Wormholes

## Traversable wormholes

- A simple example:

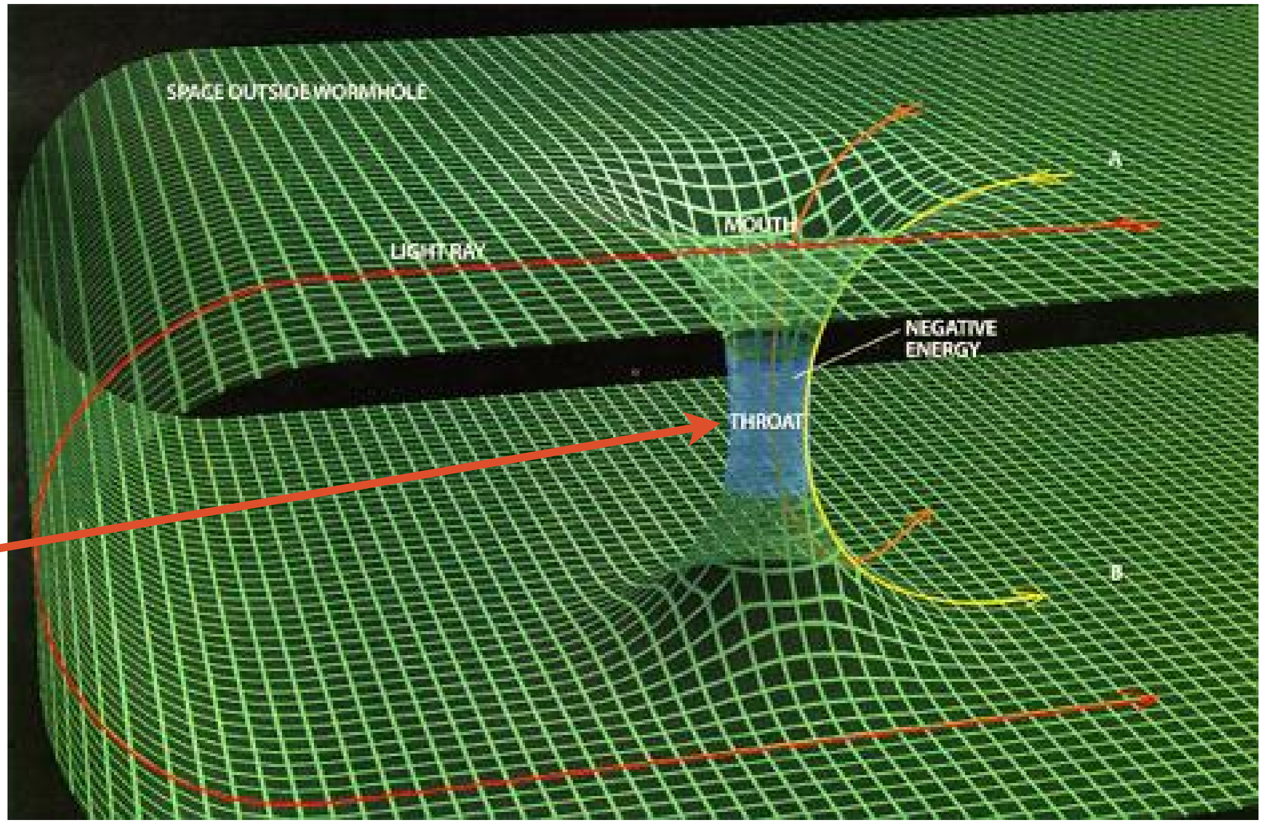
$$ds^2 = -c^2 dt^2 + d\ell^2 + (k^2+\ell^2)\big(d\theta^2 + \sin^2(\theta)\, d\varphi^2\big),$$

$$r = \pm\sqrt{k^2+\ell^2} \qquad k > 0 \text{ is a constant.}$$

- This produces the Einstein tensor

$$[G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R] = \frac{k^2}{(k^2+\ell^2)^2} \operatorname{diag}\big[-c^2, -1, (k^2+\ell^2), (k^2+\ell^2)\sin^2(\theta)\big]$$

- and so specified the energy-momentum density tensor.
- The fact that $T_{rr} < 0$ indicates that such a matter/energy distribution must be exotic.
- Keeps the throat open.



# Spacetime Engineering and Wormholes

## Traversable wormholes

- Notice that the metric was specified in terms of the square of the radial coordinate, thus making two “branches/sheets” possible.
- This can also be achieved by making the metric depend on other even functions of a radial coordinate.
- In particular, if the metric depends on $|x - x_0|$, then:
  - there will exist two “branches/sheets”
  - they meet at $x = x_0$
  - the Christoffel symbol will depend on the step-function $\vartheta(x - x_0)$
  - the Riemann tensor will depend on the Dirac delta-function $\delta(x - x_0)$
  - …as will the Ricci tensor, the scalar curvature, the Einstein tensor
  - …and so also the energy-momentum density tensor!
  - Then, the smooth part represents “bulk” matter / energy
  - the $\delta$-function part represents matter / energy localized at $x = x_0$.

# Thanks!

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