(Fundamental) Physics of Elementary Particles

Covariant derivative & the Christoffel symbol
Spacetime Curvature; Matter–gravity coupling; Special Solutions (Intro)

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Fundamental Physics of Elementary Particles

Program

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Christoffel Symbol & Covariant Derivative

Coordinate Bases

...an echo

- Basis vectors:

$$\vec{x}_\mu := (\partial_\mu \vec{r}) \quad \text{and} \quad \vec{x}^{\,\mu} := g^{\mu\nu}(\mathbf{x})\,\vec{x}_\nu,$$

- so

$$A_\mu := \vec{x}_\mu \cdot \vec{A}, \quad A^\mu := \vec{x}^{\,\mu} \cdot \vec{A}, \quad \text{and} \quad \vec{A} = A_\mu \vec{x}^{\,\mu} = A^\mu \vec{x}_\mu,$$

- and

$$\vec{x}_\mu \cdot \vec{x}_\nu = g_{\mu\nu}(\mathbf{x}) \quad \text{and} \quad \vec{x}^{\,\mu} \cdot \vec{x}^{\,\nu} = g^{\mu\nu}(\mathbf{x}).$$

- Then

$$\Gamma^\rho_{\mu\nu} : \quad (\partial_\nu \vec{x}_\mu) = \Gamma^\rho_{\mu\nu}\,\vec{x}_\rho \qquad \text{b/c basis completeness}$$

- Straightforwardly,

$$\Gamma^\rho_{\mu\nu}\vec{x}_\rho := (\partial_\mu \vec{x}_\nu) = (\partial_\mu \partial_\nu \vec{r}) = (\partial_\nu \partial_\mu \vec{r}) = (\partial_\nu \vec{x}_\mu) = \Gamma^\rho_{\nu\mu}\vec{x}_\rho.$$

- Also

$$(\partial_\mu \vec{x}^{\,\rho}) = -\Gamma^\rho_{\mu\nu}\vec{x}^{\,\nu} \quad \text{b/c} \;\; \partial_\mu\big(\vec{x}_\mu \cdot \vec{x}^{\,\nu} = \delta^\nu_\mu\big) = 0.$$

Christoffel Symbol & Covariant Derivative

Covariant Derivative

- It then follows:

$$\vec{A} := A^\rho \vec{x}_\rho \;\;\&\;\; (\partial_\mu \vec{x}_\nu) =: \Gamma^\rho_{\mu\nu} \vec{x}_\rho \;\;\Rightarrow\;\; (\partial_\mu \vec{A}) = \big[(\partial_\mu A^\rho) + \Gamma^\rho_{\mu\nu} A^\nu\big] \vec{x}_\rho;$$

$$\vec{B} := B_\rho \vec{x}^\rho \;\;\&\;\; (\partial_\mu \vec{x}^\rho) =: -\Gamma^\rho_{\mu\nu} \vec{x}^\nu \;\;\Rightarrow\;\; (\partial_\mu \vec{B}) = \big[(\partial_\mu B_\nu) - \Gamma^\rho_{\mu\nu} B_\rho\big] \vec{x}^\nu.$$

- Define:

$$D_\mu A^\rho := (\partial_\mu A^\rho) + \Gamma^\rho_{\mu\nu} A^\nu \quad \text{and} \quad D_\mu B_\nu := (\partial_\mu B_\nu) - \Gamma^\rho_{\mu\nu} B_\rho.$$

- Owing to Weyl's construction,

$$T(p,q;w) := C^w \otimes \mathcal{YS}\big[\underbrace{A \otimes \cdots \otimes A}_{p} \otimes \underbrace{B \otimes \cdots \otimes B}_{q}\big]$$

- it then ***follows*** (product rule) that:

$$(D_\mu \mathbb{T})^{\nu_1\cdots\nu_p}_{\rho_1\cdots\rho_q} = (\partial_\mu T^{\nu_1\cdots\nu_p}_{\rho_1\cdots\rho_q}) + \sum_{i=1}^{p} \Gamma^{\nu_i}_{\mu\sigma_i}\, T^{\nu_1\cdots\sigma_i\cdots\nu_p}_{\rho_1\cdots\cdots\cdots\rho_q} - \sum_{i=1}^{q} \Gamma^{\sigma_i}_{\mu\rho_i}\, T^{\nu_1\cdots\cdots\cdots\nu_p}_{\rho_1\cdots\sigma_i\cdots\rho_q}.$$

Christoffel Symbol & Covariant Derivative

Covariant Derivative

- More to the point,

$$\mathrm{X}^{\nu_1\cdots\nu_p}_{\rho_1\cdots\rho_q\,;\,\mu} := (D_\mu \mathbb{T})^{\nu_1\cdots\nu_p}_{\rho_1\cdots\rho_q}$$

- transforms as a type-(p, q+1) tensor density of weight w.
- And, since a partial derivative doesn't (verify), the Γ-symbol cannot either—so as to compensate:

$$\Gamma^{\rho}_{\mu\nu}(\mathrm{x}) = \underbrace{\frac{\partial x^\rho}{\partial y^\sigma}\frac{\partial y^\kappa}{\partial x^\mu}\frac{\partial y^\lambda}{\partial x^\nu}\Gamma^{\sigma}_{\kappa\lambda}(y)}_{\text{tensorial}} + \underbrace{\frac{\partial x^\rho}{\partial y^\sigma}\frac{\partial^2 y^\sigma}{\partial x^\mu \partial x^\nu}}_{\text{inhomogeneous}},$$

- is tensorial if and only if the transformation x → y is linear.
- In which case, no Γ_μ is needed in the first place. ☺
- True of Cartesian → Cartesian rotations & translations.

Christoffel Symbol & Covariant Derivative

Covariant Derivative

- Thus, the $\mathbb{\Gamma}_\mu$ looks awfully like a gauge potential 4-vector, except for the extra transformation matrix:

$$\mathbb{\Gamma}'_\mu = [\mathbb{U}]_\mu{}^\nu\, \mathbb{U}\, \mathbb{\Gamma}_\nu\, \mathbb{U}^{-1} + \mathbb{U}\, \partial_\mu\, \mathbb{U}^{-1}$$

- Oh, and one more thing:

$$[\mathbb{A}_\mu \cdot \Psi]^\alpha = [\mathbb{A}_\mu]_\beta{}^\alpha\, \Psi^\beta \qquad \leftrightarrow \qquad [\mathbb{\Gamma}_\mu \cdot V]^\rho = \Gamma^\rho_{\mu\nu}\, V^\nu.$$

no relation ⟷ symmetric

- This is a reflection of the conceptual non-linearity:
  - The transformation of phases is spacetime-dependent
  - The transformation of spacetime coordinates is spacetime-dependent
- Yang-Mills $\mathbb{A}_\mu$ is a spacetime 4-vector of "color"-space matrices.
- The $\mathbb{\Gamma}$-symbol is a spacetime 4-vector of spacetime matrices.

Christoffel Symbol & Covariant Derivative

Metricity of the Christoffel Symbol

- Given the relations

$$(\partial_\nu \vec{x}_\mu) = \Gamma^\rho_{\mu\nu}\, \vec{x}_\rho \quad \text{and} \quad \vec{x}_\mu \cdot \vec{x}_\nu = g_{\mu\nu}(\mathbf{x})$$

- a relation between the $\mathbb{\Gamma}$-symbol and the metric must exist. Indeed,

$$(\partial_\mu g_{\nu\rho}) = \big(\partial_\mu(\vec{x}_\nu \cdot \vec{x}_\rho)\big) = \Gamma^\sigma_{\mu\nu}\vec{x}_\sigma \cdot \vec{x}_\rho + \vec{x}_\nu \cdot \Gamma^\sigma_{\mu\rho}\vec{x}_\sigma = g_{\sigma\rho}\Gamma^\sigma_{\mu\nu} + g_{\sigma\nu}\Gamma^\sigma_{\mu\rho}$$

- produces

$$\Gamma^\rho_{\mu\nu} = \tfrac{1}{2} g^{\rho\sigma}\big[(\partial_\mu g_{\nu\sigma}) + (\partial_\nu g_{\mu\sigma}) - (\partial_\sigma g_{\mu\nu})\big] \qquad (✌)$$

- which satisfies

$$D_\mu\, g_{\nu\rho} = 0 = D_\mu\, g^{\nu\rho}. \qquad \text{covariantly constant}$$

- and vice versa: $D_\mu\, g_{\nu\rho} = 0$ with $D_\mu = \partial_\mu + \mathbb{\Gamma}_\mu$ implies Eq. (✌).
- This (Christoffel) $\mathbb{\Gamma}$-symbol is thus *metric*. adj. derived from $g_{\mu\nu}$

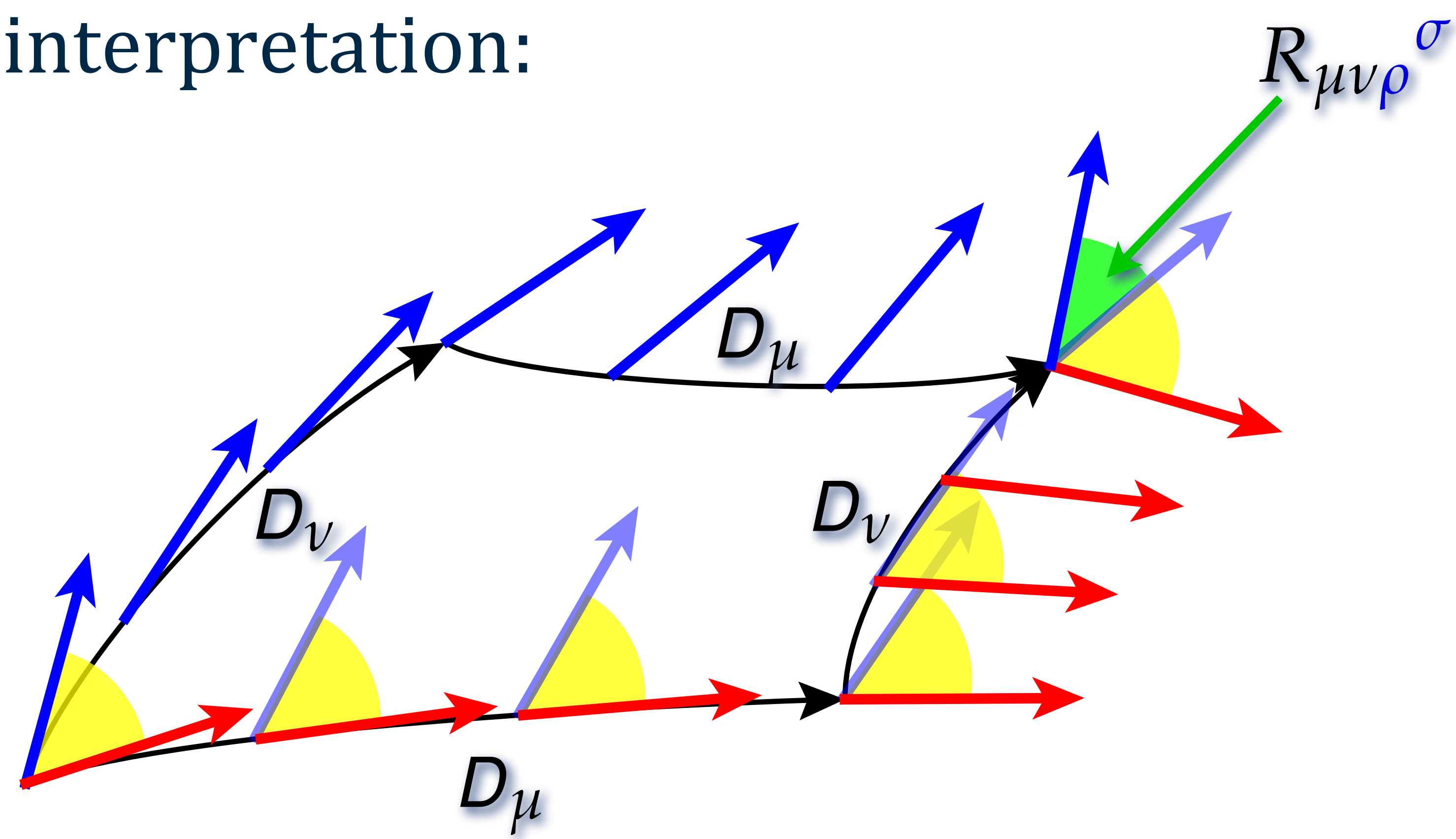
Spacetime Curvature

The Curvature Tensor

- Just like $\mathbb{F}_{\mu\nu} := \frac{\hbar c}{i g_c} [\, D_\mu \,, D_\nu \,]$
- we define

$$R_{\mu\nu\rho}{}^{\sigma} := [\, D_\mu \,, D_\nu \,]_\rho{}^{\sigma} = [(\delta^\sigma{}_\lambda \partial_\nu + \Gamma^\sigma_{\nu\lambda})\Gamma^\lambda_{\mu\rho}] - [(\delta^\sigma{}_\lambda \partial_\mu + \Gamma^\sigma_{\mu\lambda})\Gamma^\lambda_{\nu\rho}],$$
$$= \partial_\nu \Gamma^\sigma_{\mu\rho} - \partial_\mu \Gamma^\sigma_{\nu\rho} + \Gamma^\sigma_{\nu\lambda}\Gamma^\lambda_{\mu\rho} - \Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\rho}.$$

- Geometric interpretation:



Spacetime Curvature

Conditions & Contractions

- Define $R_{\mu\nu\,\rho\sigma} := R_{\mu\nu\rho}{}^{\lambda}\, g_{\lambda\sigma}$ (no such thing for $\mathbb{F}_{\mu\nu}$)
- The Riemann tensor satisfies the following identities:

$R_{\mu\nu\rho}{}^{\rho} = 0,$	(non-abelian)	$\mathrm{Tr}[\mathbb{F}_{\mu\nu}] = 0$
$R_{\mu\nu\,\rho\sigma} = -R_{\nu\mu\,\rho\sigma},$		$\mathbb{F}_{\mu\nu} = -\mathbb{F}_{\nu\mu}$
$R_{\mu\nu\,\rho\sigma} = -R_{\mu\nu\,\sigma\rho},$		—
$R_{\mu\nu\,\rho\sigma} = +R_{\rho\sigma\,\mu\nu},$		—
$\varepsilon^{\lambda\nu\rho\sigma} R_{\mu\nu\,\rho\sigma} = 0,$	1st Bianchi identity	—
$\varepsilon^{\kappa\lambda\mu\nu} D_{\lambda} R_{\mu\nu\,\rho\sigma} = 0.$	2nd Bianchi identity	$\varepsilon^{\kappa\lambda\mu\nu} D_{\lambda}\mathbb{F}_{\mu\nu} = 0$

- The Riemann tensor is part 1st derivative, part quadratic in $\mathbb{\Gamma}_{\mu}$
- ...just as $\mathbb{F}_{\mu\nu}$ is part 1st derivative, part quadratic in $\mathbb{A}_{\mu}$
- ...of 2nd order in derivatives of the metric, $g_{\mu\nu}$, & homogeneous!
  It also involves $g^{\mu\nu}$, which is very non-linear in $g_{\mu\nu}$!

Spacetime Curvature

Conditions & Contractions

- For the Yang-Mills type field strength tensor,

$$g^{\mu\nu}\mathbb{F}_{\mu\nu} \equiv 0, \qquad \begin{cases} \mathrm{Tr}[\mathbb{F}_{\mu\nu}] = [\mathbb{F}_{\mu\nu}]_\alpha{}^\alpha = 0, & \text{for semisimple Lie groups,} \\ \mathrm{Tr}[F_{\mu\nu}] = F_{\mu\nu}, & \text{for } U(1) \text{ factors,} \end{cases}$$

- Since all four indices in $R_{\mu\nu\rho}{}^\sigma$ are of the same type, we *can* define:

$$\text{Ricci tensor:} \qquad R_{\mu\rho} := R_{\mu\nu\rho}{}^\nu,$$

invariant

$$\text{scalar curvature:} \qquad R := g^{\mu\rho} R_{\mu\rho} = g^{\mu\rho} R_{\mu\nu\rho}{}^\nu.$$

- It is then possible to define:
  - $S_{\mu\nu\rho\sigma}$, the "pure trace" part, = $\frac{1}{12} R(g_{\mu\rho}\, g_{\nu\sigma} - g_{\mu\sigma}\, g_{\nu\rho})$.
  - $E_{\mu\nu\rho\sigma}$, the "semi-traceless" part, = $(g_{\mu[\rho}\, S_{\nu]\sigma} - g_{\nu[\rho}\, S_{\sigma]\mu})$; $S_{\mu\nu} := R_{\mu\nu} - \frac{1}{4}\, g_{\mu\nu} R$.
  - $C_{\mu\nu\rho\sigma}$, the fully traceless part, Weyl (conformal curvature) tensor.
- Also:

invariant

$$\|R_{\mu\nu}\|^2 := R_{\mu\nu}\, g^{\mu\rho} g^{\nu\sigma}\, R_{\rho\sigma}$$

invariant

$$\|R_{\mu\nu\rho}{}^\sigma\|^2 := R_{\mu\nu\rho}{}^\sigma\, g^{\mu\alpha} g^{\nu\beta} g^{\rho\gamma} g_{\sigma\delta}\, R_{\alpha\beta\gamma}{}^\delta$$

Spacetime Curvature

The Einstein-Hilbert Action

- For the Yang-Mills case, the only way to construct a Lagrangian density quadratic in $\mathbb{F}_{\mu\nu}$ is $\propto \mathrm{Tr}[\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu}]$.
- By the same token, consider:

quadratic in *R*

$$\int \sqrt{-g}\, \mathrm{d}^4x\; R_{\mu\nu\rho}{}^{\sigma}\, g^{\mu\kappa} g^{\nu\lambda}\, R_{\kappa\lambda\sigma}{}^{\rho}.$$

  - Varying w.r.t. components of Γ_μ produces a 2nd order PDE for Γ_μ
  - Varying w.r.t. components of $g_{\mu\nu}$ produces a 4th order PDE for $g_{\mu\nu}$
  - Unlike with Yang-Mills $\mathbb{F}_{\mu\nu}$, we now do have *R*, so:

$$\frac{c^3}{16\pi\, G_N} \int \sqrt{-g}\, \mathrm{d}^4x\; R,$$

linear in *R*

  - is the Einstein-Hilbert action.
    - So that the units are ML^2/T, where $[\mathrm{d}^4x] = 4$ and $[g_{\mu\nu}] = 0$
    - Varying w.r.t. components of $g_{\mu\nu}$ produces a 2nd order PDE for $g_{\mu\nu}$.

Matter–Gravity Coupling

"Covariantizing" Lagrangians

- Varying the Einstein-Hilbert action produces

$$G_{\mu\nu} := R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 0.$$

- This is the 2nd order PDE of motion for $g_{\mu\nu}$. **Empty spacetime!**
- $R_{\mu\nu\rho}{}^{\sigma}$ and $R_{\mu\nu}$ and R are all (very) nonlinear in $g_{\mu\nu}$, this is a highly non-trivial, nonlinear PDE system.
- Coupling everything else to this gauge-GCT theory:

$$S[\phi_i(\mathbf{x})] = \int \mathrm{d}^4 x\, \mathscr{L}\big(\phi_i, (\partial_\mu \phi_i), \cdots; \mathbf{x}; C_a\big)$$

$$\to \int \sqrt{|g|}\, \mathrm{d}^4 x \left[\frac{c^3}{16\pi G_N} R - \mathscr{L}\big(\phi_i, (D_\mu \phi_i), \cdots; \mathbf{x}; C_a\big) \right]$$

any and all non-metric/Christoffel fields

Matter–Gravity Coupling

Einstein Equations

- Varying the GCT-covariantized action w.r.t. $g_{\mu\nu}$ produces

Einstein equations: $$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \frac{8\pi\, G_N}{c^4} T_{\mu\nu},$$

- where

Energy-momentum: $$T_{\mu\nu} := -\frac{2c}{\sqrt{-g}} \frac{\delta\big(\sqrt{-g}\,\mathscr{L}_M\big)}{\delta g^{\mu\nu}}$$

- So, the presence of matter curves spacetime.
  - T_{00}: energy density
  - $T_{0i} = T_{i0}$: linear momentum density
  - $T_{ik} = T_{ki}$ $(i \neq k)$: shear stresses
  - T_{ii} (no sum): normal stresses, called "pressure" if all are equal

$$T^{\mu\nu} := g^{\mu\rho} T_{\rho\sigma} g^{\nu\sigma}$$

$$D_\mu T^{\mu\nu} = 0$$
continuity equation

Noether Thm.

Matter–Gravity Coupling

Two Oblique Parallels

- By construction,

$$[\mathbb{A}_\mu]_\alpha{}^\beta \quad\longleftrightarrow\quad \Gamma^\rho_{\mu\nu},$$

$$[\mathbb{F}_{\mu\nu}]_\alpha{}^\beta \quad\longleftrightarrow\quad R_{\mu\nu\,\rho}{}^\sigma$$

$$\cup \qquad\qquad \cup$$

$$\vec{\mathbb{E}} = (\mathbb{F}_{0i}),\ \vec{\mathbb{B}} = (\mathbb{F}_{ij}) \qquad C_{\mu\nu\,\rho}{}^\sigma, E_{\mu\nu\,\rho}{}^\sigma, S_{\mu\nu\,\rho}{}^\sigma$$

not very useful because all indices mix!

- While $(\mathbb{F}_{0i})$ and $(\mathbb{F}_{ij})$ indeed are irreducible representations of $SO(0,3)\times G_{\rm YM}$ (*i.e.*, rotations × gauge group),
- $(\mathbb{R}_{0i})$ and $(\mathbb{R}_{ij})$ are irreducible representations of neither $SO(0,3)$ (rotations) nor $SO(1,3)$ (full Lorentz group).
- Although $(\mathbb{A}_\mu \leftrightarrow \mathbb{\Gamma}_\mu)$ and $(\mathbb{F}_{\mu\nu} \leftrightarrow \mathbb{R}_{\mu\nu})$ are conceptually analogous, this analogy has technical limitations.
- Unh… ¿?

Matter–Gravity Coupling

Two Oblique Parallels

- On the other hand…
- The Einstein equations

$$\Big\{R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\,R = \tfrac{1}{2}g^{\rho\sigma}(\partial_\mu\partial_\rho g_{\nu\sigma} + \partial_\nu\partial_\rho g_{\mu\sigma}) + \ldots\Big\} = \frac{8\pi\, G_N}{c^4}T_{\mu\nu}$$

- remind awfully much of Gauss-Ampère equations

$$\Big\{(\Box A^\mu) - \eta^{\mu\nu}(\partial_\nu\partial_\rho A^\rho)\Big\} = \frac{1}{4\pi\epsilon_0}\,\frac{4\pi}{c}\,j_e^\nu\,.$$

- So,

$j_e^\mu \quad\longleftrightarrow\quad T_{\mu\nu},$
both are Noether currents

$A_\mu \quad\longleftrightarrow\quad g_{\mu\nu},$
both are “most basic” fields

- Just as every 4-current produces an EM field
- & every EM field specifies the 4-current it needs to support it,
- so are the energy-momentum tensor and spacetime curvature linked and shalt not be rendered asunder.

Matter–Gravity Coupling

Two Oblique Parallels

- To summarize:

EM/YM	GCT	
	conceptually	engineeringly
—	$g_{\mu\nu}$	—
$\mathbb{A}_\mu$	$\mathbb{\Gamma}_\mu$	$g_{\mu\nu}$
$\mathbb{F}_{\mu\nu}$	$\mathbb{R}_{\mu\nu}$	$\mathbb{\Gamma}_\mu$
$\mathbb{J}_\mu$	?	$T_{\mu\nu}$

Special Solutions (Intro)

Special Solutions: Intro

A Quick Trick...

- Consider the Einstein equations:

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \frac{8\pi G_N}{c^4} T_{\mu\nu},$$

- ...the trace of which equates

$$R - \tfrac{1}{2} 4R = \frac{8\pi G_N}{c^4} g^{\mu\nu} T_{\mu\nu}, \quad i.e., \quad R = -\frac{8\pi G_N}{c^4} g^{\mu\nu} T_{\mu\nu}$$

- whereby the Einstein equations are equivalent to

$$R_{\mu\nu} = \frac{8\pi G_N}{c^4} \Big[T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} \big(g^{\rho\sigma} T_{\rho\sigma} \big) \Big]$$

This is ***not*** the traceless part of the energy-momentum tensor!

- So,

$$\big(R_{\mu\nu} = 0 \big) \iff \big(T_{\mu\nu} = 0 \big)$$

Ricci-flatness

- Ricci-flat spacetimes require/imply no material support
- Absence of matter implies/requires Ricci-flat spacetimes

Special Solutions: Intro

A Quick Trick...

- Why is "Ricci-flatness" so important?
- Well, construct $\boldsymbol{R} := \mathrm{d}x^{\mu}\mathrm{d}x^{\nu}R_{\mu\nu}$. This is a 2-form.
- Taken modulo total derivatives, this defines the 1st Chern class.
- Integrals over 2-dimensional submanifolds X are invariants of continuous deformations of X, within the spacetime
- More importantly, $\boldsymbol{R}\wedge\boldsymbol{R} = \mathrm{d}^4x\ \varepsilon^{\mu\nu\rho\sigma}R_{\mu\nu}R_{\rho\sigma}$ is a 4-form
- ...and may be integrated over the whole spacetime manifold
- ...and is a topological invariant (1st Chern number, C_1) of the whole spacetime manifold.
- Ricci-flatness implies that C_1(spacetime) = 0.

I'LL BE BACK.

Special Solutions: Intro

Immaterial (Ricci-flat) Solutions

- Consider empty space.
- That is, space with no matter. (immaterial)
- In 1915, Karl Schwatzschild, while at the Russian front as a German soldier, found the first and best-known Ricci-flat solution to Einstein's equations. He died within a year.

$$[g_{\mu\nu}] = \mathrm{diag}\Big(-f_S(r), \tfrac{1}{f_S(r)}, r^2, r^2\sin^2(\theta)\Big),$$

$$\mathrm{d}s^2 = -f_S(r)\,c^2\mathrm{d}t^2 + \tfrac{1}{f_S(r)}\mathrm{d}r^2 + r^2\big(\mathrm{d}\theta^2 + \sin^2(\theta)\,\mathrm{d}\varphi^2\big),$$

$$f_S(r) := \Big(1 - \frac{r_S}{r}\Big), \qquad r_S = \frac{2G_N\,M}{c^2}.$$

- But, if there was no matter to begin with, whose mass is M?
- It is the mass of the singularity—a "defect" in spacetime—at the origin.

Empty spacetime can have mass, even classically!

Special Solutions: Intro

Immaterial (Ricci-flat) Solutions

- Singularity??

$$[g_{\mu\nu}] \;=\text{diag}\Big(-f_S(r), \tfrac{1}{f_S(r)}, r^2, r^2\sin^2(\theta)\Big) \qquad f_S(r) := \Big(1-\frac{r_S}{r}\Big)$$

- At both $r = r_S$ and $r = 0$, a metric component blows up.
  - At $r = r_S$, $f_S(r) = 0$, the dt^2-term vanishes & the dr^2-terms blows up.
  - At $r = 0$, $f_S(r) = \infty$, the dr^2-term vanishes & the dt^2-terms blows up.
- But, that may well be an artifact of “bad” coordinates! Metric components are not invariants; they form a type-(0,2) tensor!
- Indeed, in 1933, Georges Lemaître realized that a coordinate system introduced by Arthur Eddington in 1924 proves that the $r = r_S$ location is perfectly uneventful.
- In turn, the Kretschmann curvature invariant is

$$\|R_{\mu\nu\rho}{}^{\sigma}\|^2 = \frac{48{G_N}^2\,M^2}{c^4\,r^6}$$

@ $r = 0$:
Ka-blooey!
(a technical term)

Special Solutions: Intro

Immaterial (Ricci-flat) Solutions

- Unh… "the $r = r_S$ location is perfectly uneventful" is a bit of an understatement.
- Actually, something does happen there:

$$v_1 = \sqrt{\frac{2G_N M}{r}}.$$

- is the "escape speed" from a gravitational source of mass M.

$$r_S = \frac{2G_N M}{c^2} \quad \Rightarrow \quad M = \frac{c^2 r_S}{2G_N} \quad \Rightarrow \quad v_1 = \sqrt{\frac{2G_N \frac{c^2 r_S}{2G_N}}{r}} = c\sqrt{\frac{r_S}{r}}$$

- …so the "escape speed" becomes unattainable. Event horizon.

Location of no-return.

- Oh, and one more thing! Within the event horizon,

$$ds^2 = +|f_S(r)|\,c^2 dt^2 - \frac{1}{|f_S(r)|}dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)\,d\varphi^2\right)$$

the physical meaning of r & t is swapped.

Special Solutions: Intro

Immaterial (Ricci-flat) Solutions

- When discussing Yang-Mills (EM, Strong, Weak) interactions, we assumed a flat, $\mathbb{R}^{1,3}$-like spacetime. Even the "topologically non-trivial" solutions do not change the spacetime. It's an *arena*.
- In general relativity, non-trivial spacetimes are not $\mathbb{R}^{1,3}$-like.
- In so-modeling gravity, we *can* excise portions of spacetime
- ...though that may render the spacetime somehow *incomplete*.
- Spacetime (non-)singularity may well thus be a subtle issue.
  - **Geodesically complete**; refine: time-like, null, space-like.
  - **Metrically complete**: convergence of all Cauchy sequences.
  - **B-complete**: if every C^1-curve of finite length is contained.
  - **Curvature invariants**: $R_{\mu\nu\rho}{}^{\sigma}$ has 20 independent DoF's; no known list.
- B-completeness implies geodesic completeness, and coincides with metric completeness—only for $g_{\mu\nu} \geq 0$, not for spacetime.

Thanks!

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