(Fundamental) Physics of Elementary Particles

Quantum Electrodynamics of Leptons: Feynman Rules and Renormalization

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Fundamental Physics of Elementary Particles

Program

- Quantum Electrodynamics Calculations
  - The Lagrangian and classical field theory
  - Feynman rules: fundamental processes
  - …and their combinations
- Effective Cross-Sections and Lifetimes
  - Mott and Rutherford scattering
  - Electron-positron annihilation/creation
- Renormalization
  - A computation
  - …and its physical meaning
  - The renormalization "group"
  - Effective action & partition functional

Quantum Electrodynamics Calculations

The Lagrangian and classical field theory

- Recall: $\partial_\mu \to D_\mu := \partial_\mu + \frac{i}{\hbar c} A_\mu\, Q,$

$$\mathscr{L}_{QED} = \overline{\Psi}(\mathrm{x})\,[i\hbar c\,\not{D} - mc^2]\,\Psi(\mathrm{x}) - \frac{4\pi\epsilon_0}{4} F_{\mu\nu}F^{\mu\nu}, \qquad \not{D} := \boldsymbol{\gamma}^\mu D_\mu,$$

$$= \overline{\Psi}(\mathrm{x}) \Big[\boldsymbol{\gamma}^\mu (\hbar c\, i\partial_\mu - q_\Psi A_\mu) - mc^2\Big]\,\Psi(\mathrm{x})$$

$$- \frac{4\pi\epsilon_0}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)\eta^{\mu\rho}\eta^{\nu\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho).$$

- To obtain (classical) equations of motion, *vary*.

$$\frac{\delta}{\delta A_\rho(\mathrm{x})} \int \mathrm{d}^4\mathrm{y}\ \mathscr{F}\big(A_\mu(\mathrm{y}), (\partial_\mu A_\nu(\mathrm{y}))\big)$$

$$= \int \mathrm{d}^4\mathrm{y}\ \frac{\delta A_\sigma(\mathrm{y})}{\delta A_\rho(\mathrm{x})}\,\frac{\delta}{\delta A_\sigma(\mathrm{y})}\mathscr{F}\big(A_\mu(\mathrm{y}), (\partial_\mu A_\nu(\mathrm{y}))\big),$$

$$= \int \mathrm{d}^4\mathrm{y}\ \delta^4(\mathrm{x}-\mathrm{y})\,\delta_\rho^\sigma \frac{\partial}{\partial A_\sigma(\mathrm{y})}\mathscr{F}\big(A_\mu(\mathrm{y}), (\partial_\mu A_\nu(\mathrm{y}))\big),$$

$$= \frac{\partial}{\partial A_\rho(\mathrm{x})}\mathscr{F}\big(A_\mu(\mathrm{x}), (\partial_\mu A_\nu(\mathrm{x}))\big).$$

Quantum Electrodynamics Calculations

The Lagrangian and classical field theory

- Using thus

$$\frac{\partial}{\partial A_\rho(\mathrm{x})} A_\mu(\mathrm{x}) = \delta^\rho_\mu, \qquad \frac{\partial}{\partial A_\rho(\mathrm{x})}\left(\partial_\mu A_\nu(\mathrm{x})\right) = 0,$$

$$\frac{\partial}{\partial(\partial_\rho A_\sigma(\mathrm{x}))} A_\mu(\mathrm{x}) = 0, \qquad \frac{\partial}{\partial(\partial_\rho A_\sigma(\mathrm{x}))}\left((\partial_\mu A_\nu(\mathrm{x})\right) = \delta^{\rho\sigma}_{\mu\nu} := \delta^\rho_\mu \delta^\sigma_\nu,$$

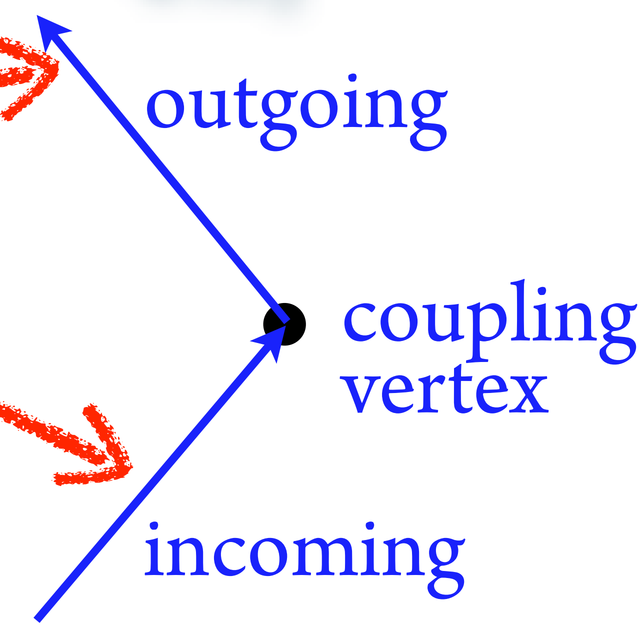
- we obtain

$$\partial_\mu \frac{\partial \mathscr{L}_{QED}}{\partial(\partial_\mu A_\nu)} = \frac{\partial \mathscr{L}_{QED}}{\partial A_\nu} \qquad \Rightarrow \qquad \partial_\mu F^{\mu\nu} = \frac{q_\Psi}{4\pi\epsilon_0}\overline{\Psi}\boldsymbol{\gamma}^\nu\Psi,$$

- and

$$\frac{q_\Psi c}{4\pi}\overline{\Psi}\boldsymbol{\gamma}^\mu\Psi =: j_e^\mu$$

- is the electric current 4-vector density.



Quantum Electrodynamics Calculations

The Lagrangian and classical field theory

- So, varying with respect to A_μ and $\overline{\Psi}$ (from the left),

$$\partial_\mu F^{\mu\nu} = \frac{q_\Psi}{4\pi\epsilon_0}\overline{\Psi}\boldsymbol{\gamma}^\nu\Psi, \qquad \Big[i\hbar c\,\boldsymbol{\gamma}^\mu\partial_\mu - mc^2\mathbb{1}\Big]\Psi = q_\Psi A_\mu\boldsymbol{\gamma}^\mu\Psi.$$

- A little fermionic digression:
- Algebra: $[f,g]=0,\ \ [f,\psi]=[f,\chi]=0=[g,\psi]=[g,\chi],$ but $\{\psi,\chi\}=0.$
- Calculus:

$$\Big[\frac{\partial}{\partial f},\frac{\partial}{\partial g}\Big]=0,\ \ \Big[\frac{\partial}{\partial f},\frac{\partial}{\partial\psi}\Big]=\Big[\frac{\partial}{\partial f},\frac{\partial}{\partial\chi}\Big]=0=\Big[\frac{\partial}{\partial g},\frac{\partial}{\partial\psi}\Big]=\Big[\frac{\partial}{\partial g},\frac{\partial}{\partial\chi}\Big],\ \text{but}\ \Big\{\frac{\partial}{\partial\psi},\frac{\partial}{\partial\chi}\Big\}=0.$$

- … so

$$\frac{\partial}{\partial\psi}\chi\cdots = -\chi\frac{\partial}{\partial\psi}\cdots \quad\text{and}\quad \frac{\partial}{\partial\chi}\psi\cdots = -\psi\frac{\partial}{\partial\chi}\cdots,$$

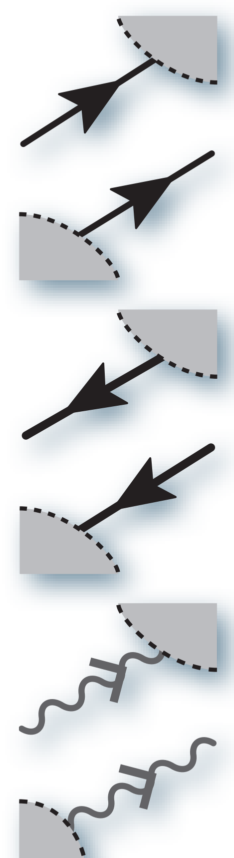
- Alternatively:

$$\psi\frac{\overleftarrow{\partial}}{\partial\psi}=1,\quad (\psi\chi)\frac{\overleftarrow{\partial}}{\partial\psi}=-\Big(\psi\frac{\overleftarrow{\partial}}{\partial\psi}\Big)\chi=-\chi,\ \ (\psi\chi)\frac{\overleftarrow{\partial}}{\partial\chi}=\psi\Big(\chi\frac{\overleftarrow{\partial}}{\partial\chi}\Big)=\psi,\ \ \textit{etc.}$$

Quantum Electrodynamics Calculations

Feynman rules: fundamental processes

- 4-momenta & polarizations
  - Denote in/out-coming 4-momenta by p_1, p_2, …
    Denote internal 4-momenta by q_1, q_2, …
  - For spin-½ particles, orient lines along with 4-momenta, oppositely for spin-½ antiparticles.
  - Polarizations:

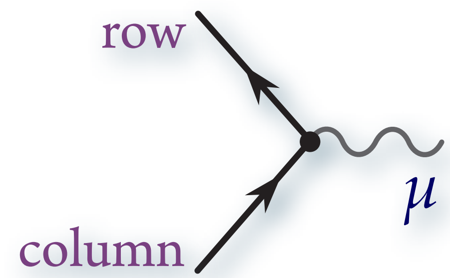


Spin-½ particle	incoming	u^s	$s =$ spin projection $= \uparrow, \downarrow$
	outgoing	$\overline{u}_s$	
Spin-½ antiparticle	incoming	$\overline{v}_s$	($\cong$ spin-½ particle, travels backwards in time)
	outgoing	v^s	
Photon	incoming	ϵ^μ	$\epsilon^\mu p_\mu = 0$ and $\epsilon^0 = 0$
	outgoing	$\epsilon^{\mu *}$	

Quantum Electrodynamics Calculations

Feynman rules: fundamental processes

- Vertices:



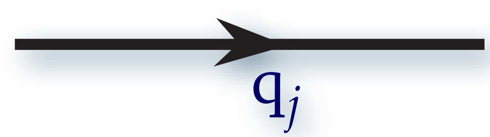
$$\rightarrow \quad -ig_e\boldsymbol{\gamma}^\mu$$

matrix

$$\mathscr{L}_{\rm QED} = \cdots - \overline{\Psi}_A\,(\boldsymbol{\gamma}^\mu)^A{}_B(q_\Psi A_\mu)\Psi^B + \ldots$$

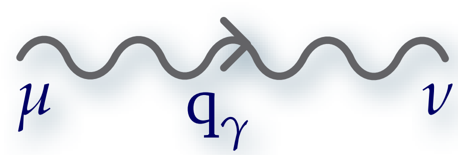
- Internal propagations = lines:

spin-½ particle:



$$\rightarrow \quad \frac{i}{\not{q}_j - m_j c} = i\frac{\not{q}_j + m_j c\mathbb{1}}{\mathrm{q}_j^2 - m_j^2 c^2},$$

photon:



$$\rightarrow \quad -i\frac{\eta_{\mu\nu}}{\mathrm{q}_\gamma^2}$$

  - …which are ***off-shell***.
- 4-momentum conservation:
  - assign each vertex a factor $(2\pi)^4\delta^4(\sum_j \mathrm{k}_j)$,
  - …just like in Kirchoff rules for electrical circuits.

Quantum Electrodynamics Calculations

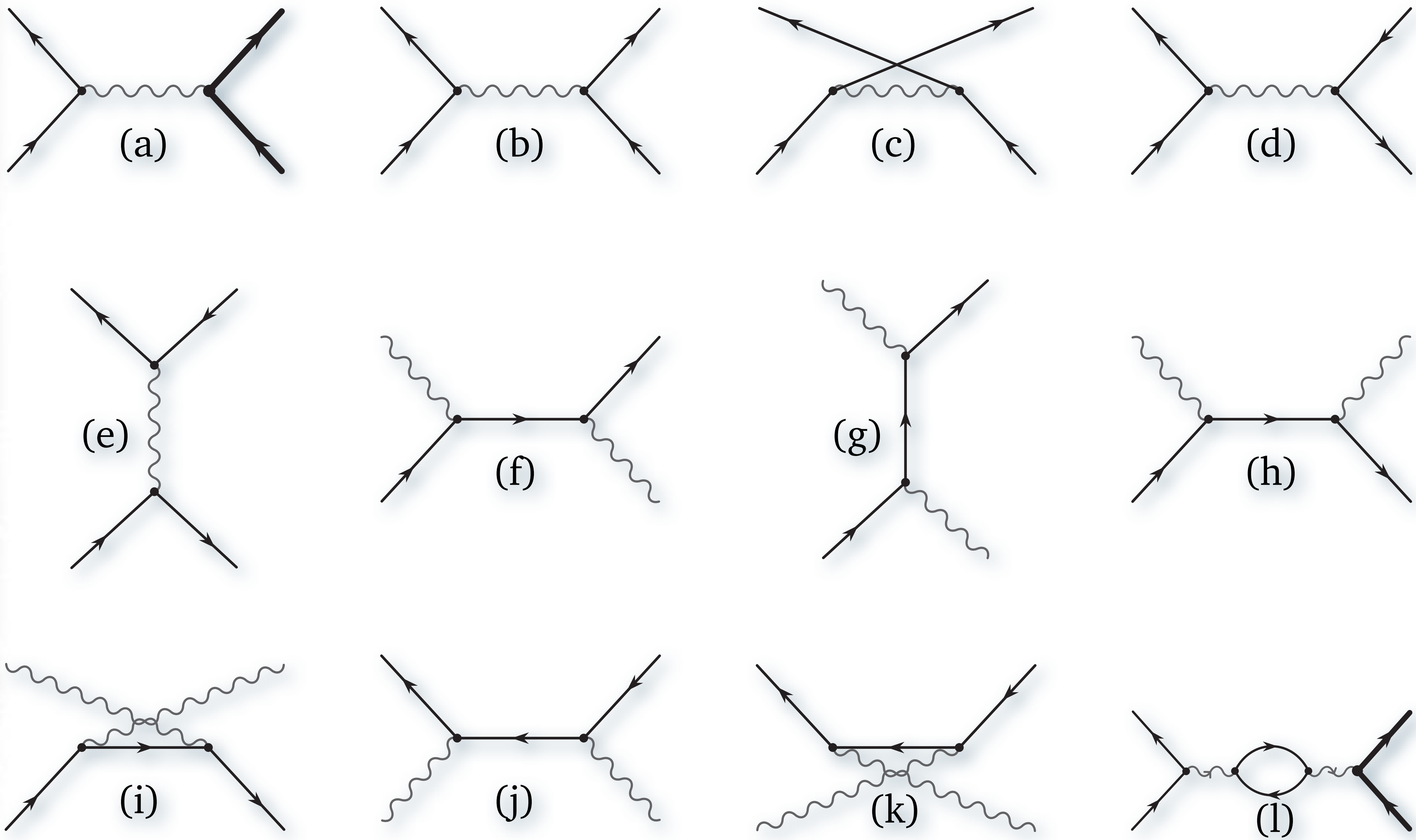
Feynman rules: fundamental processes

- Integrate over $(2\pi)^{-4}\, d^4 q_j$, for all internal 4-momenta.
  - Notice, the δ-functions at each vertex are used to eliminate some of the integrations over internal 4-momenta. Do the math!
- Closed internal (virtual) fermion loops incur "$\times(-1)$" each.
- The result equals $-i\,\mathfrak{M}\,(2\pi)^4\,\delta^4(\Sigma_j\, p_j)$,
  - …from which we read off the matrix element, $\mathfrak{M}$.
- All amplitudes that contribute to the same **process**,
  - (as defined by incoming and outgoing particle states)
  - are summed to produce the total amplitude of the process.
- If the Feynman graphs for two amplitudes, $\mathfrak{M}_1$ and $\mathfrak{M}_2$, differ by the swap of two identical fermions,
  - they must be added with a relative sign -1.

Quantum Electrodynamics Calculations

Combinations of Fundamental Processes

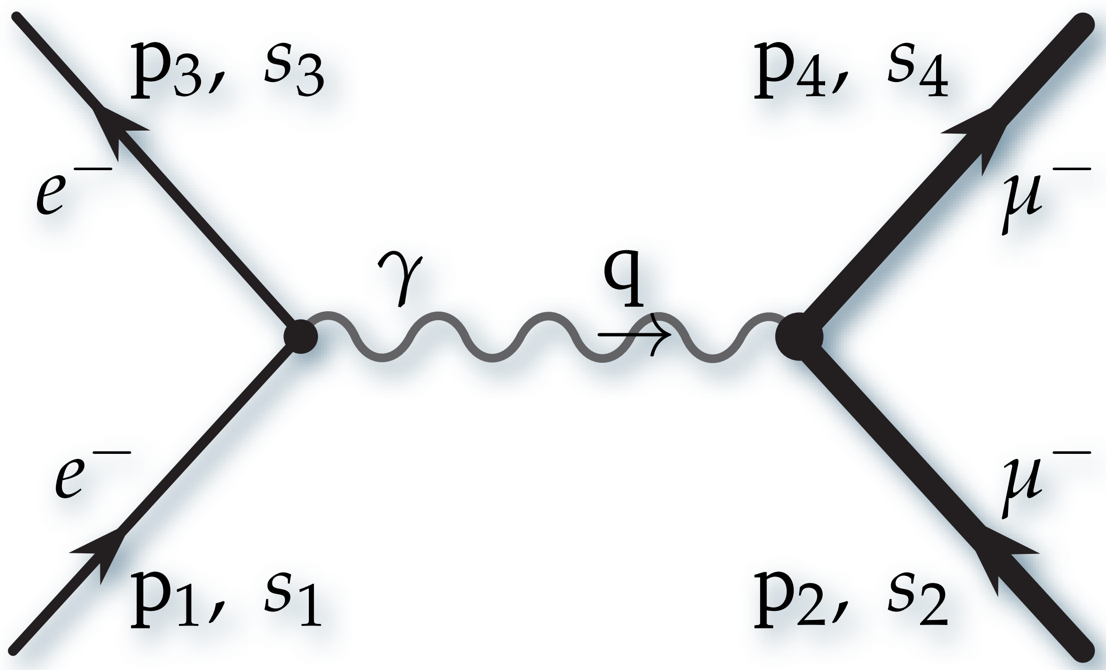
- Twelve Apostles:



Quantum Electrodynamics Calculations

Combinations of Fundamental Processes

- 1st:



$$\int \frac{d^4q}{(2\pi)^4}(2\pi)^4\delta^4(p_1 - p_3 - q)\,(2\pi)^4\delta^4(p_2 - p_4 + q)$$

$$\left[\overline{u^{s_3}}_A(p_3)(ig_e\,\gamma^{\mu A}{}_B)u^{s_1,B}(p_1)\right]\left(\frac{-i\eta_{\mu\nu}}{q^2}\right)\left[\overline{U^{s_4}}_C(p_4)(ig_e\,\gamma^{\nu C}{}_D)U^{s_2,D}(p_2)\right]$$

- Using the 1st δ-function, sets $q = p_1 - p_3$, and removes the integration (but not the factors of 2π).
- This turns the 2nd δ-function into $\delta^4(p_1+p_2-p_3-p_4)$.

Quantum Electrodynamics Calculations

Combinations of Fundamental Processes

- A bit of simplifying yields

$$\frac{ig_e^2(2\pi)^4}{(\mathrm{p}_1-\mathrm{p}_3)^2}\,\delta^4(\mathrm{p}_2-\mathrm{p}_4+\mathrm{p}_1-\mathrm{p}_3)$$
$$\left[\overline{u^{s_3}}_A(\mathrm{p}_3)\,\gamma^{\mu A}{}_B\,u^{s_1,B}(\mathrm{p}_1)\right]\left[\overline{U^{s_4}}_C(\mathrm{p}_4)\,\gamma_\mu{}^C{}_D\,U^{s_2,D}(\mathrm{p}_2)\right],$$

- which lets us identify

$$\mathfrak{M}_{(a)}=-\frac{g_e^2}{(\mathrm{p}_1-\mathrm{p}_3)^2}\left[\overline{u^{s_3}}_A(\mathrm{p}_3)\,\gamma^{\mu A}{}_B\,u^{s_1,B}(\mathrm{p}_1)\right]\left[\overline{U^{s_4}}_C(\mathrm{p}_4)\,\gamma_\mu{}^C{}_D\,U^{s_2,D}(\mathrm{p}_2)\right].$$

- If the spins (s_j) of the in/out particles are known, insert them, and use the Dirac spinors and Dirac matrices as given before.
- If they are not known/measured, one must sum over all possible contributions.
- Note, however, that we need $|\mathfrak{M}|^2$, not $\mathfrak{M}$ alone.

Quantum Electrodynamics Calculations

Combinations of Fundamental Processes

- But, $\mathfrak{M}^\dagger$ contains the conjugate factor:

$$\begin{aligned}
\left[\overline{u}_A(\mathrm{p}_3)\,\gamma^{\mu A}{}_B\,u^B(\mathrm{p}_1)\right]^\dagger &= \left[u^\dagger(\mathrm{p}_3)\,\boldsymbol{\gamma}^0\,\boldsymbol{\gamma}^\mu\,u(\mathrm{p}_1)\right]^\dagger = \left[u^\dagger(\mathrm{p}_1)\,(\boldsymbol{\gamma}^\mu)^\dagger\,(\boldsymbol{\gamma}^0)^\dagger\,u(\mathrm{p}_3)\right],\\
&= \left[u^\dagger(\mathrm{p}_1)\mathbb{1}\,(\boldsymbol{\gamma}^\mu)^\dagger\,\boldsymbol{\gamma}^0\,u(\mathrm{p}_3)\right] = \left[u^\dagger(\mathrm{p}_1)\boldsymbol{\gamma}^0\,\boldsymbol{\gamma}^0\,(\boldsymbol{\gamma}^\mu)^\dagger\,\boldsymbol{\gamma}^0\,u(\mathrm{p}_3)\right],\\
&= \left[\overline{u}(\mathrm{p}_1)\,\overline{\boldsymbol{\gamma}}^\mu\,u(\mathrm{p}_3)\right], \qquad \overline{\boldsymbol{\gamma}}^\mu := \boldsymbol{\gamma}^0(\boldsymbol{\gamma}^\mu)^\dagger\boldsymbol{\gamma}^0,
\end{aligned}$$

- …so (the sum over spins of) $|\mathfrak{M}|^2$ contains the factor:

$$\begin{aligned}
&\sum_{s_1,s_3}\left[\overline{u^{s_3}}_A(\mathrm{p}_3)\,\gamma^{\mu A}{}_B\,u^{s_1 B}(\mathrm{p}_1)\right]\left[\overline{u^{s_1}}_C(\mathrm{p}_1)\,\overline{\gamma}^{\nu C}{}_D\,u^{s_3 D}(\mathrm{p}_3)\right],\\
&= \mathrm{Tr}\left[\boldsymbol{\gamma}^\mu\,(\not{p}_1 + m_e c\mathbb{1})\,\overline{\boldsymbol{\gamma}}^\nu\,(\not{p}_3 + m_e c\mathbb{1})\right],
\end{aligned}$$

- …which expands into a multinomial in the components of the 4-momenta p_1 and p_3.
- The same is then done for the p_2-p_4 (muon) part…

Quantum Electrodynamics Calculations

Combinations of Fundamental Processes

- ...obtaining:

$$\langle|\mathfrak{M}_{(a)}|^2\rangle = \frac{g_e^4}{(\mathrm{p}_1-\mathrm{p}_3)^4}\sum_{s_1,s_3}\mathrm{Tr}\left[\overline{u^{s_3}}(\mathrm{p}_3)\,\boldsymbol{\gamma}^\mu\,u^{s_1}(\mathrm{p}_1)\right]\mathrm{Tr}\left[\overline{u^{s_1}}(\mathrm{p}_1)\,\overline{\boldsymbol{\gamma}}^\nu\,u^{s_3}(\mathrm{p}_3)\right]$$

$$\times\sum_{s_2,s_4}\mathrm{Tr}\left[\overline{U^{s_4}}(\mathrm{p}_4)\,\boldsymbol{\gamma}_\mu\,U^{s_2}(\mathrm{p}_2)\right]\mathrm{Tr}\left[\overline{U^{s_2}}(\mathrm{p}_2)\,\overline{\boldsymbol{\gamma}}_\nu\,U^{s_4}(\mathrm{p}_4)\right],$$

$$= \frac{g_e^4}{(\mathrm{p}_1-\mathrm{p}_3)^4}\,\boxed{\mathrm{Tr}\left[\boldsymbol{\gamma}^\mu\,(\not{p}_1+m_e c\mathbb{1})\,\overline{\boldsymbol{\gamma}}^\nu\,(\not{p}_3+m_e c\mathbb{1})\right]}$$

$$\times\,\mathrm{Tr}\left[\boldsymbol{\gamma}_\mu\,(\not{p}_2+m_\mu c\mathbb{1})\,\overline{\boldsymbol{\gamma}}_\nu\,(\not{p}_4+m_\mu c\mathbb{1})\right],$$

$$= \frac{g_e^4}{(\mathrm{p}_1-\mathrm{p}_3)^4}\,\boxed{X^{\mu\nu}(1,3;e^-)}\,X_{\mu\nu}(2,4;\mu^-).$$

$$\langle|\mathfrak{M}_{(a)}|^2\rangle = \frac{8g_e^4}{(\mathrm{p}_1-\mathrm{p}_3)^4}\Big[(\mathrm{p}_1\cdot\mathrm{p}_2)(\mathrm{p}_3\cdot\mathrm{p}_4)+(\mathrm{p}_1\cdot\mathrm{p}_4)(\mathrm{p}_3\cdot\mathrm{p}_2)+2(m_e m_\mu c^2)^2$$

$$-(m_\mu c)^2(\mathrm{p}_1\cdot\mathrm{p}_3)-(m_e c)^2(\mathrm{p}_2\cdot\mathrm{p}_4)\Big].$$

Quantum Electrodynamics Calculations

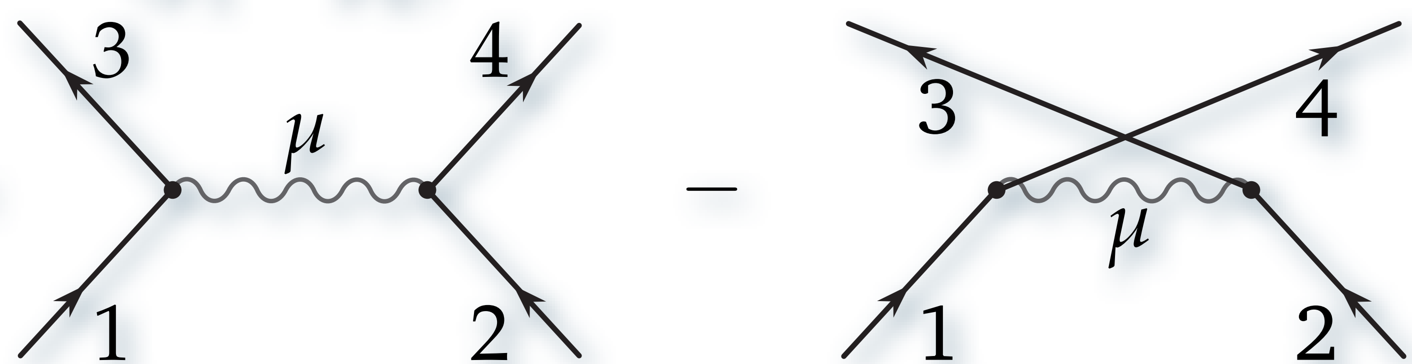
Combinations of Fundamental Processes

- Adapt this for e^--e^- scattering:

$$\mathfrak{M}_{2e^-\to 2e^-} = \mathfrak{M}_{(b)} - \mathfrak{M}_{(c)}$$

$$= -\frac{g_e^2}{(\mathrm{p}_1-\mathrm{p}_3)^2}\left[\overline{u_3}\,\boldsymbol{\gamma}^\mu\,u_1\right]\left[\overline{u_4}\,\boldsymbol{\gamma}_\mu\,u_2\right]$$

$$+\frac{g_e^2}{(\mathrm{p}_1-\mathrm{p}_4)^2}\left[\overline{u_4}\,\boldsymbol{\gamma}^\mu\,u_1\right]\left[\overline{u_3}\,\boldsymbol{\gamma}_\mu\,u_2\right],$$

$=$  $-$

$$|\mathfrak{M}_{2e^-\to 2e^-}|^2 = |\mathfrak{M}_{(b)}|^2 + |\mathfrak{M}_{(c)}|^2 - 2\,\Re e\left(\mathfrak{M}^\dagger_{(b)}\,\mathfrak{M}_{(c)}\right).$$

$$\mathfrak{M}^\dagger_{(b)}\mathfrak{M}_{(c)} \propto [\overline{u}_2\overline{\boldsymbol{\gamma}}^\mu u_4][\overline{u}_1\overline{\boldsymbol{\gamma}}_\mu u_3][\overline{u}_4\boldsymbol{\gamma}^\nu u_1][\overline{u}_3\boldsymbol{\gamma}_\nu u_2],$$

$$= \left[\overline{u}_2\overline{\boldsymbol{\gamma}}^\mu u_4\overline{u}_4\boldsymbol{\gamma}^\nu u_1\overline{u}_1\overline{\boldsymbol{\gamma}}_\mu u_3\overline{u}_3\boldsymbol{\gamma}_\nu u_2\right],$$

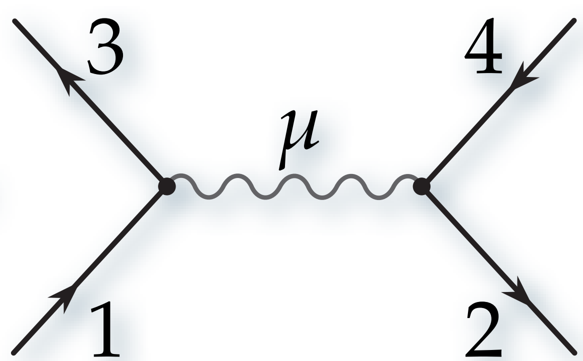
may be summed over spins as before

Quantum Electrodynamics Calculations

Combinations of Fundamental Processes

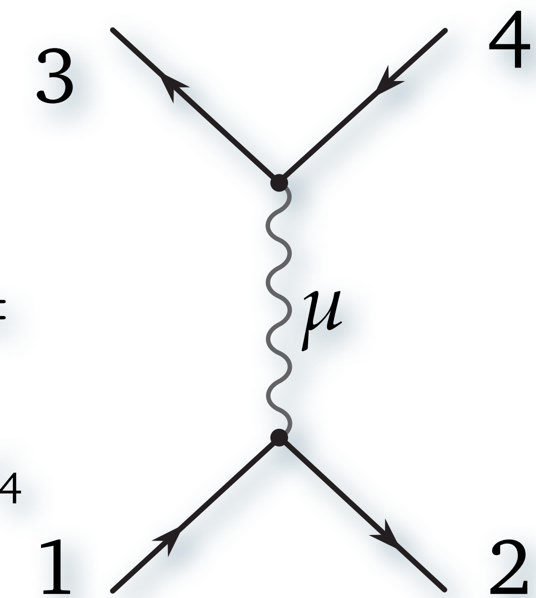
- Adapt this for e^--e^+ scattering:

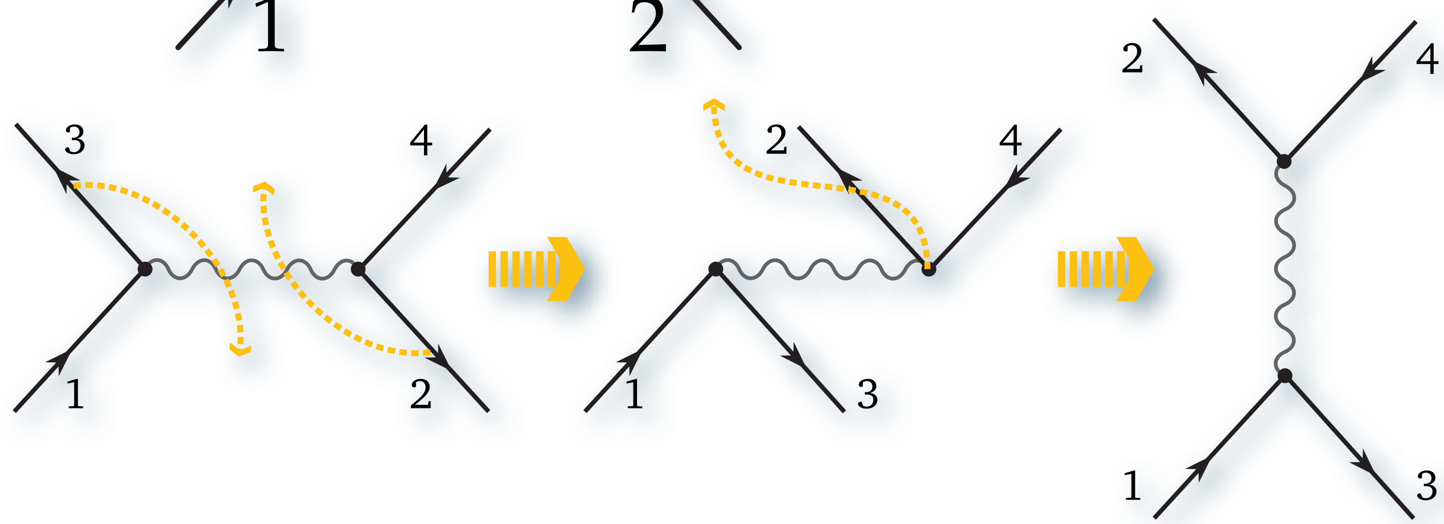
$$\mathfrak{M}_{(d)} =$$



and

$$\mathfrak{M}_{(e)} =$$





$$\begin{aligned}\mathfrak{M}_{e^-e^+\to e^-e^+} &= \mathfrak{M}_{(d)} - \mathfrak{M}_{(e)},\\ &= -\frac{g_e^2}{(\mathrm{p}_1-\mathrm{p}_3)^2}[\overline{u}_3\boldsymbol{\gamma}^\mu u_1][\overline{v}_2\boldsymbol{\gamma}_\mu v_4] + \frac{g_e^2}{(\mathrm{p}_1+\mathrm{p}_2)^2}[\overline{v}_2\boldsymbol{\gamma}^\mu u_1][\overline{u}_3\boldsymbol{\gamma}_\mu v_4],\end{aligned}$$

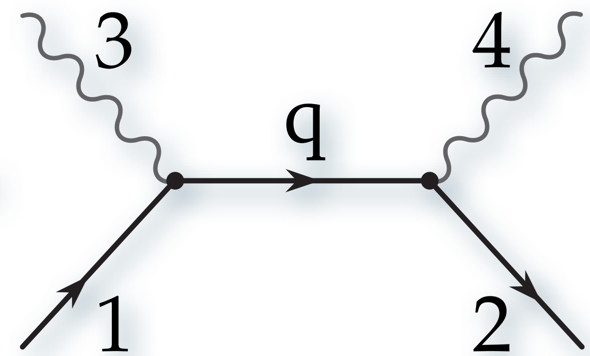
- …and so on and so forth…

Quantum Electrodynamics Calculations

Combinations of Fundamental Processes

- Adapt this for e^--e^+ annihilation:

$$\mathfrak{M}_{e^- + e^+ + \to 2\gamma} = \mathfrak{M}_{(h)} + \mathfrak{M}_{(i)} =$$



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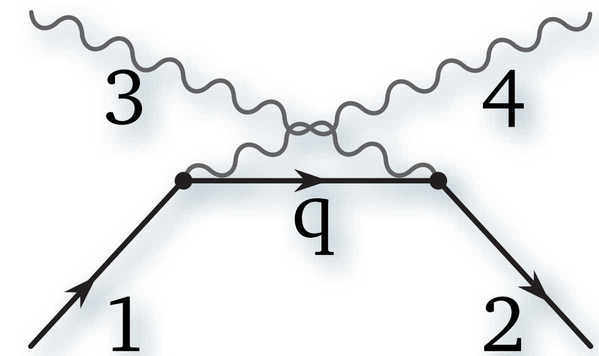
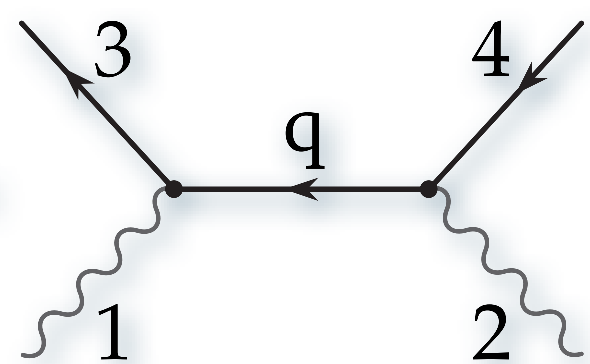


- …or e^--e^+ pair creation:

$$\mathfrak{M}_{2\gamma \to e^- + e^+} =$$



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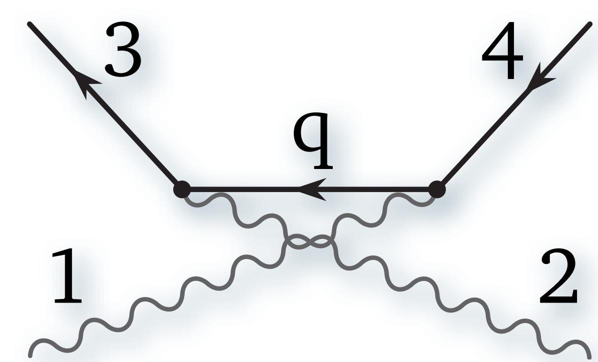


- This makes it evident that

$$\mathfrak{M}_{2\gamma \to e^- + e^+} = \mathfrak{M}^{\dagger}_{e^- + e^+ \to 2\gamma}.$$

Effective Cross-Sections and Lifetimes

Mott and Rutherford scattering

- For scattering experiments, work in the "2"-rest frame (target), with "1" the "probe."
- Unlike in QM, however, the target *does* move after scattering, although we may look into the approximation where the target recoil is neglected.
- Now, you did do the homework problems; right?
- Right? Riiight??
- … ahem …
- And, you recall examples 1.3 (for 2-particle decay) and 1.4 (for a 2-particle scattering)?
- So, you remember:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S\,|\mathfrak{M}|^2}{(E_A + E_B)^2} \frac{|\vec{p}_f|}{|\vec{p}_i|}.$$

Effective Cross-Sections and Lifetimes

Mott and Rutherford scattering

- OK, now that we… ahem… refreshed our memories…
- Let me ask again:
- You did do the homework problems; right? Riiight??

✎ **1.3.3** *For the elastic collision $A + B \to A' + B'$, in a system where B is originally at rest (and is the target), derive:*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \approx S\left(\frac{\hbar}{8\pi}\right)^2 \frac{|\mathfrak{M}|^2}{m_B} \frac{\vec{p}_{A'}^{\,2}}{|\vec{p}_A|\left|\left(|\vec{p}_{A'}|(E_A + m_B c^2) - |\vec{p}_A| E_{A'} \cos\theta\right)\right|}.$$

✎ **1.3.4** *Show that the result of the previous problem simplifies when $(m_A/m_B) \ll 1$:*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \approx S\left(\frac{\hbar\, E_{A'}}{8\pi\, E_A}\right)^2 \frac{|\mathfrak{M}|^2}{m_B^2}.$$

Effective Cross-Sections and Lifetimes

Mott and Rutherford scattering

- Furthermore,

  ✎ ***1.3.5*** *For the elastic collision in exercise 1.3.3 but in the case when the recoil of the target after the collision may be neglected since $m_B c^2 \gg E_A$, derive:*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \approx \left(\frac{\hbar}{8\pi m_B c}\right)^2 |\mathfrak{M}|^2.$$

- …and:

  ✎ ***1.3.6*** *For the inelastic collision $A + B \to C_1 + C_2$, in a system where B (the target) is originally at rest, and $(m_{C_i}/m_A) \ll 1$ and $(m_{C_i}/m_B) \ll 1$, derive:*

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \approx \left(\frac{\hbar}{8\pi}\right)^2 \frac{S\,|\mathfrak{M}|^2}{m_B(E_A + m_B c^2 - |\vec{p}_A|c\,\cos\theta)}\frac{|\vec{p}_{C_1}|}{|\vec{p}_A|},$$

where θ is the angle between $\vec{p}_1$ and $\vec{p}_3$.

Effective Cross-Sections and Lifetimes

Mott and Rutherford scattering

- Sooo…
- For the scattering of e^- on p^+ (or on $\mu^\pm$, at a stretch),

$$\frac{d\sigma}{d\Omega} \approx \left(\frac{\hbar}{8\pi M c}\right)^2 \left\langle |\mathfrak{M}|^2 \right\rangle$$

$$\underset{\text{probe}}{\mathrm{p}_1 = (E/c, \vec{p}_1)}, \qquad \underset{\text{target}}{\mathrm{p}_2 = (Mc, \vec{0})}, \qquad \underset{\text{probe, scattered}}{\mathrm{p}_3 \approx (E/c, \vec{p}_3)}, \qquad \underset{\text{no recoil}}{\mathrm{p}_4 \approx (Mc, \vec{0})}$$

$$(\mathrm{p}_1 - \mathrm{p}_3)^2 \approx -(\vec{p}_1 - \vec{p}_3)^2 = -\vec{p}_1^{\,2} - \vec{p}_3^{\,2} + 2\vec{p}_1 \cdot \vec{p}_3 = -4\vec{p}^2 \sin^2(\theta/2),$$

$$(\mathrm{p}_1 \cdot \mathrm{p}_3) \approx \frac{E^2}{c^2} - \vec{p}_1 \cdot \vec{p}_3 = \vec{p}^2 + m^2c^2 - \vec{p}^2 \cos\theta = m^2c^2 + 2\vec{p}^2 \sin^2(\theta/2),$$

$$(\mathrm{p}_1 \cdot \mathrm{p}_2) = ME \approx (\mathrm{p}_2 \cdot \mathrm{p}_3) \approx (\mathrm{p}_1 \cdot \mathrm{p}_4) \approx (\mathrm{p}_3 \cdot \mathrm{p}_4), \qquad (\mathrm{p}_2 \cdot \mathrm{p}_4) \approx M^2c^2.$$

- …producing:

$$\left\langle |\mathfrak{M}|^2 \right\rangle \approx \left(\frac{g_e^2\, Mc}{\vec{p}^{\,2}\, \sin^2(\theta/2)}\right)^2 \left(m^2c^2 + \vec{p}^{\,2} \cos^2(\theta/2)\right),$$

Effective Cross-Sections and Lifetimes

Mott and Rutherford scattering

- Mott's formula:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \approx \left(\frac{\alpha\hbar}{2\,\vec{p}^{\,2}\sin^2(\theta/_2)}\right)^2 \left(m^2c^2 + \vec{p}^{\,2}\cos^2(\theta/_2)\right)$$

- In the approximation where $\vec{p}^2 \ll m^2c^2$, this reduces to:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \approx \left(\frac{\alpha\hbar}{2\,\vec{p}^{\,2}\sin^2(\theta/_2)}\right)^2 m^2c^2 = \left(\frac{\alpha\hbar c}{2\,m\vec{v}^{\,2}\sin^2(\theta/_2)}\right)^2,$$

- …the classical Rutherford formula.
- Recall the approximations:
  - $m_A \ll m_B$,
  - target recoil neglected,
  - non-relativistic linear momenta.

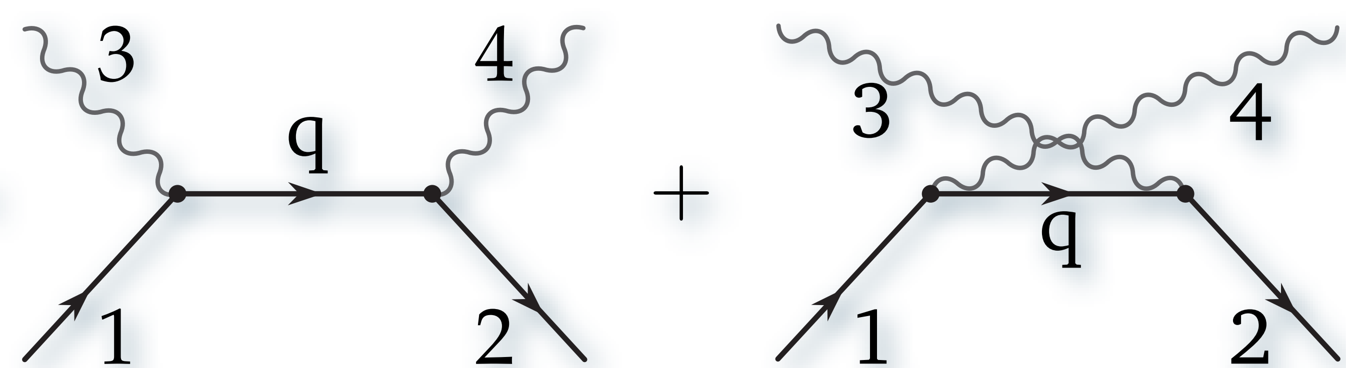
Yet, this modest result helped discovering the atomic nucleus!

Effective Cross-Sections and Lifetimes

Electron-positron annihilation

- The positronium decay into two photons (why not one?) may be described as an inelastic scattering $e^- + e^+ \longrightarrow 2\gamma$.
- The electron and the positron have to be at the same place, and it is known that they move slowly; KE $\ll mc^2$ …
  - … so, we'll approximate them as static: $p_1 = p_2 = m_e c\,(1, 0, 0, 0)$.
  - The photons, in turn, are far from static: $p_3 = p_4 = m_e c\,(1, 0, 0, \pm 1)$.
  - Photon polarization vectors:
    - Lorenz gauge: $\boldsymbol{\varepsilon}_3\cdot\boldsymbol{p}_3 = 0 = \boldsymbol{\varepsilon}_4\cdot\boldsymbol{p}_4$.
    - Coulomb gauge: $\boldsymbol{\varepsilon}_3\cdot\boldsymbol{p}_1 = 0 = \boldsymbol{\varepsilon}_4\cdot\boldsymbol{p}_1$ and $\boldsymbol{\varepsilon}_3\cdot\boldsymbol{p}_2 = 0 = \boldsymbol{\varepsilon}_4\cdot\boldsymbol{p}_2$.

$$\mathfrak{M}_{e^- + e^+ + \rightarrow 2\gamma} = \mathfrak{M}_{(h)} + \mathfrak{M}_{(i)} =$$



Effective Cross-Sections and Lifetimes

Electron-positron annihilation

- For the amplitudes, we get:

$$\mathfrak{M}_{(h)} = \frac{g_e^2}{(\mathrm{p}_1 - \mathrm{p}_3)^2 - m_e^2 c^2}\big([\overline{v}_2 \not{\epsilon}_4^* (\not{p}_1 - \not{p}_3 + m_e c\mathbb{1})\not{\epsilon}_3^* u_1]\big),$$

$$\not{\epsilon}_i^* := \epsilon_i^{*\,\mu}\boldsymbol{\gamma}_\mu$$

$$\mathfrak{M}_{(i)} = \frac{g_e^2}{(\mathrm{p}_1 - \mathrm{p}_4)^2 - m_e^2 c^2}\big([\overline{v}_2 \not{\epsilon}_3^* (\not{p}_1 - \not{p}_4 + m_e c\mathbb{1})\not{\epsilon}_4^* u_1]\big),$$

$$\not{p}_1\not{\epsilon}_3^* = -\not{\epsilon}_3^*\not{p}_1 + 2\epsilon_3^*\cdot\mathrm{p}_1 = -\not{\epsilon}_3^*\not{p}_1, \quad \not{p}_3\not{\epsilon}_3^* = -\not{\epsilon}_3^*\not{p}_3 + 2\epsilon_3^*\cdot\mathrm{p}_3 = -\not{\epsilon}_3^*\not{p}_3,$$

$$(\not{p}_1 - \not{p}_3 + m_e c\mathbb{1})\not{\epsilon}_3^* u_1 = \not{\epsilon}_3^*(-\not{p}_1 + \not{p}_3 + m_e c\mathbb{1})u_1 = \not{\epsilon}_3^*\not{p}_3 u_1,$$

since $(\not{p}_1 - m_e c\mathbb{1})u_1 = 0.$

$$\mathfrak{M}_{e^- + e^+ \to 2\gamma} = -\frac{g_e^2}{2m_e c}\overline{v}_2[\not{\epsilon}_4^*\not{\epsilon}_3^*(\boldsymbol{\gamma}^0 + \boldsymbol{\gamma}^3) + \not{\epsilon}_3^*\not{\epsilon}_4^*(\boldsymbol{\gamma}^0 - \boldsymbol{\gamma}^3)]u_1,$$

$$= -\frac{g_e^2}{2m_e c}\overline{v}_2[-2\vec{\epsilon}_4^*\cdot\vec{\epsilon}_3^*\boldsymbol{\gamma}^0 + i(\vec{\epsilon}_4^*\times\vec{\epsilon}_3^*)\cdot\vec{\boldsymbol{\Sigma}}\boldsymbol{\gamma}^3]u_1,$$

$$\boldsymbol{\Sigma}_i := 2\varepsilon_{ijk}\,\boldsymbol{\gamma}^{jk} = \tfrac{i}{2}\,\varepsilon_{ijk}\,[\boldsymbol{\gamma}^j, \boldsymbol{\gamma}^k].$$

Effective Cross-Sections and Lifetimes

Electron-positron annihilation

- Using the Dirac basis of Dirac matrices,

$$\mathfrak{M}_{\uparrow\downarrow} = -2ig_e^2(\vec{\epsilon}_3^{\,*} \times \vec{\epsilon}_4^{\,*})_z = -\mathfrak{M}_{\downarrow\uparrow},$$

- … and the e^-+e^+-system had to have been in the "singlet" state, antisymmetric in the spins: $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$.
- Choose $\varepsilon_3 = (0, -1, -i, 0)/\sqrt{2}$ and $\varepsilon_4 = (0, 1, -i, 0)/\sqrt{2}$

$$(\vec{\epsilon}_3^{\,*} \times \vec{\epsilon}_4^{\,*})_{\uparrow\downarrow} = \left(\vec{\epsilon}^{\,*}_{3,|1,+1\rangle} \times \vec{\epsilon}^{\,*}_{4,|1,-1\rangle}\right) = -\frac{1}{2}\begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ 1 & -i & 0 \\ 1 & i & 0 \end{vmatrix} = -i\,\hat{e}_3$$

- and the photons's spins must be antisymmetrized too.
- Putting all this together yields:

$$\mathfrak{M}_{e^-+e^+\to 2\gamma} = -4g_e^2.$$

And

& antiparallel spins

Effective Cross-Sections and Lifetimes

Electron-positron annihilation

- The differential cross section is then:

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi(E_1 + E_2)}\right)^2 \frac{|\vec{p}_f|}{|\vec{p}_i|} |\mathfrak{M}|^2 = \left(\frac{\hbar c}{16\pi(m_e c)}\right)^2 \frac{|E_\gamma / c|}{|m_e v|} \left| -4 g_e^2 \right|^2,$$

$$E_1 = mc^2 = E_2 \text{ and } E_\gamma = m_e c^2.$$

- Simplifying,

$$\frac{d\sigma}{d\Omega} = \frac{1}{cv}\left(\frac{\hbar\alpha}{m_e}\right)^2, \qquad \sigma = \frac{4\pi}{cv}\left(\frac{\hbar\alpha}{m_e}\right)^2.$$

- Except, for a decay, we need a decay constant:

$$\Gamma = v\,\sigma\,|\Psi(\vec{0},t)|^2 \qquad |\Psi(\vec{0},t)|^2 = \left(\frac{\alpha m_e c}{\hbar\, n}\right)^3$$

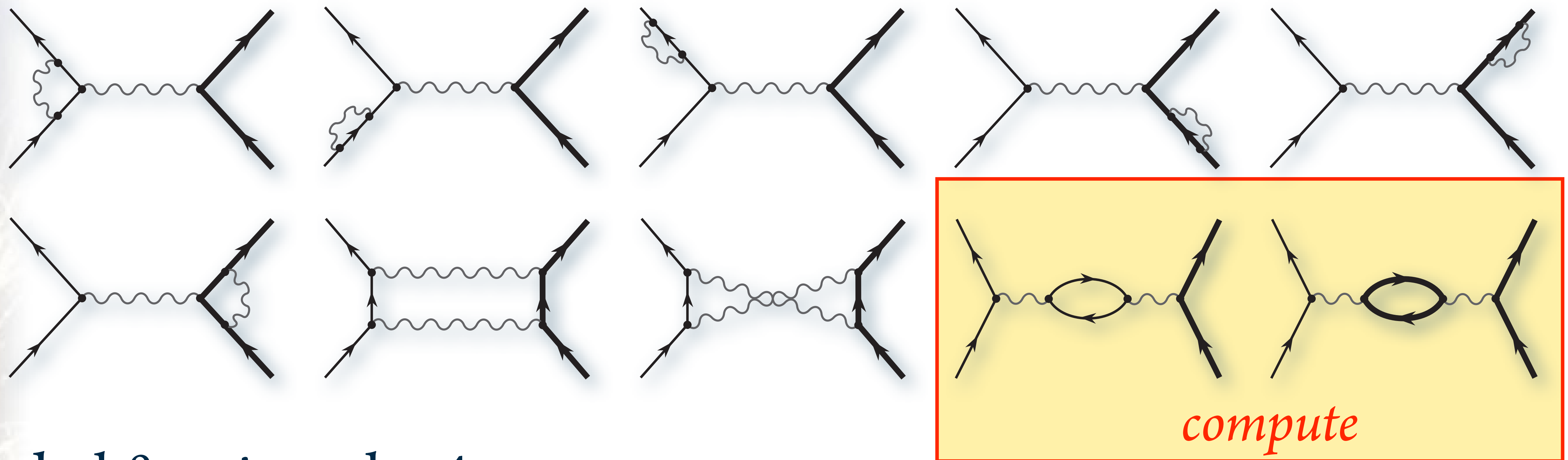
$$\Gamma = \frac{4\pi}{c}\left(\frac{\hbar\alpha}{m_e}\right)^2 \left[\frac{1}{\pi}\left(\frac{\alpha(\frac{1}{2}m_e)c}{\hbar\, n}\right)^3\right] = \frac{\alpha^5\, m_e c^2}{2\,\hbar\, n^3},$$

$$\tau = \frac{1}{\Gamma} = \frac{2\,\hbar\, n^3}{\alpha^5\, m_e c^2} \approx (1.24494 \times 10^{-10}\,\text{s}) \times n^3.$$

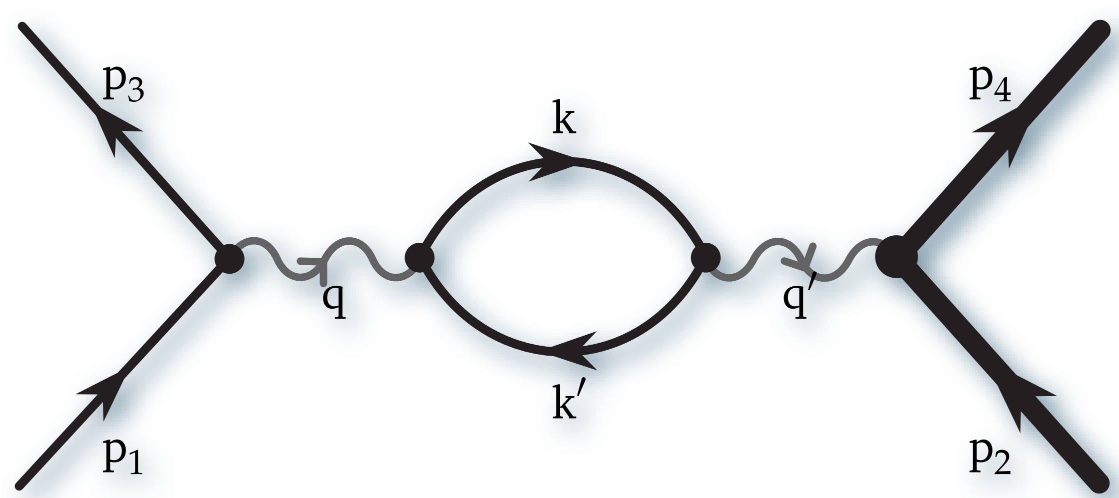
Renormalization

A computation...

- Consider the $O(g^4)$ corrections to $e^-+\mu^-$-scattering:



- Label & orient the 4-momenta:



…and then see how the result depends on the mass of p_2-p_4.

Renormalization

A computation...

- Following the procedure:

$$\int \frac{d^4q}{(2\pi)^4}\frac{d^4q'}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{d^4k'}{(2\pi)^4}$$

$$\times (2\pi)^4\delta^4(p_1-p_3-q)\,(2\pi)^4\delta^4(q-k+k')\,(2\pi)^4\delta^4(k-k'-q')$$

$$\times (2\pi)^4\delta^4(p_2-p_4+q')\,[\overline{u}_3(ig_e\boldsymbol{\gamma}^\mu)u_1]\left(\frac{-i\eta_{\mu\nu}}{q^2}\right)$$

$$\times (-1)\,\mathrm{Tr}\left[(ig_e\boldsymbol{\gamma}^\nu)\frac{i}{\not{k}-m_ec}(ig_e\boldsymbol{\gamma}^\rho)\frac{i}{\not{k}-m_ec}\right]\left(\frac{-i\eta_{\rho\sigma}}{(q')^2}\right)[\overline{U}_4(ig_e\boldsymbol{\gamma}^\sigma)U_2],$$

$$= -i(2\pi)^4\,\delta^4(p_1+p_2-p_3-p_4)$$

$$\times\left[\frac{-ig_e^4}{q^4}\int\frac{d^4k}{(2\pi)^4}[\overline{u}_3\boldsymbol{\gamma}^\mu u_1][\overline{U}_4\boldsymbol{\gamma}^\rho U_2]\right.$$

$$\left.\times\frac{\mathrm{Tr}[\boldsymbol{\gamma}_\mu(\not{k}+m_ec)\boldsymbol{\gamma}_\rho(\not{k}-\not{q}+m_ec)]}{(k^2-m_e^2c^2)[(k-q)^2-m_e^2c^2]}\right]_{q=p_1-p_3}$$

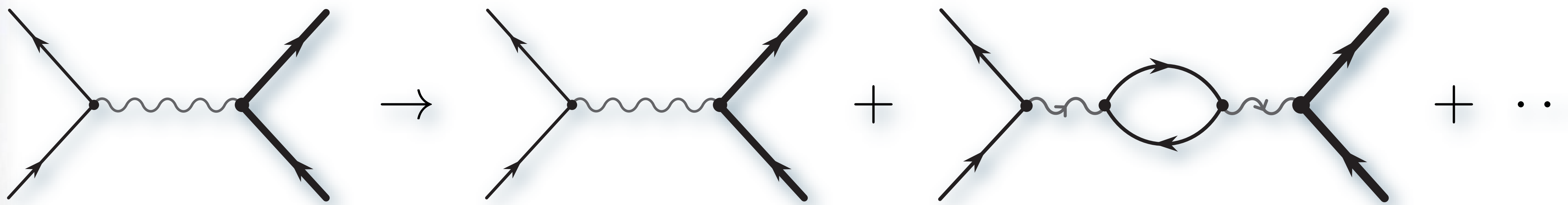
$\mathfrak{M}$

Note that one integration is left, not having been eliminated.

Renormalization

A computation...

- This changes the photon propagator (internal line factor):



$$\mathfrak{M}_{(a)} \quad \to \quad \mathfrak{M}_{(a;2)} + \mathfrak{M}_{(a';4)} + \cdots$$

$$\frac{-i\eta_{\mu\rho}}{\mathrm{q}^2} \quad \to \quad \frac{-i\eta_{\mu\rho}}{\mathrm{q}^2} + \frac{-i\,H_{\mu\rho}}{\mathrm{q}^4} + \cdots \quad = \quad \frac{-i}{\mathrm{q}^2}\Big[\eta_{\mu\rho} + \frac{H_{\mu\rho}}{\mathrm{q}^2} + \cdots\Big],$$

$$H_{\mu\rho} \quad := \quad i g_e^2 \int \frac{\mathrm{d}^4\mathrm{k}}{(2\pi)^4} \frac{\mathrm{Tr}[\boldsymbol{\gamma}_\mu(\not{k} + m_e c)\boldsymbol{\gamma}_\rho(\not{k} - \not{q} + m_e c)]}{(\mathrm{k}^2 - m_e^2 c^2)[(\mathrm{k}-\mathrm{q})^2 - m_e^2 c^2]}.$$

- It behooves us to use the fact that

$$H_{\mu\rho} = -\eta_{\mu\rho}\,\mathrm{q}^2\, I(\mathrm{q}^2) + q_\mu\, q_\rho\, J(\mathrm{q}^2),$$

- & drop $J(\mathrm{q}^2)$: in $\mathfrak{M}$, $[\overline{u}_3 \boldsymbol{\gamma}^\mu\, u_1] q_\mu = [\overline{u}_3\,(\not{p}_1 - \not{p}_3)\, u_1] = 0.$

on-shell: $\not{p}_1\, u_1 = m_e c\, u_1$

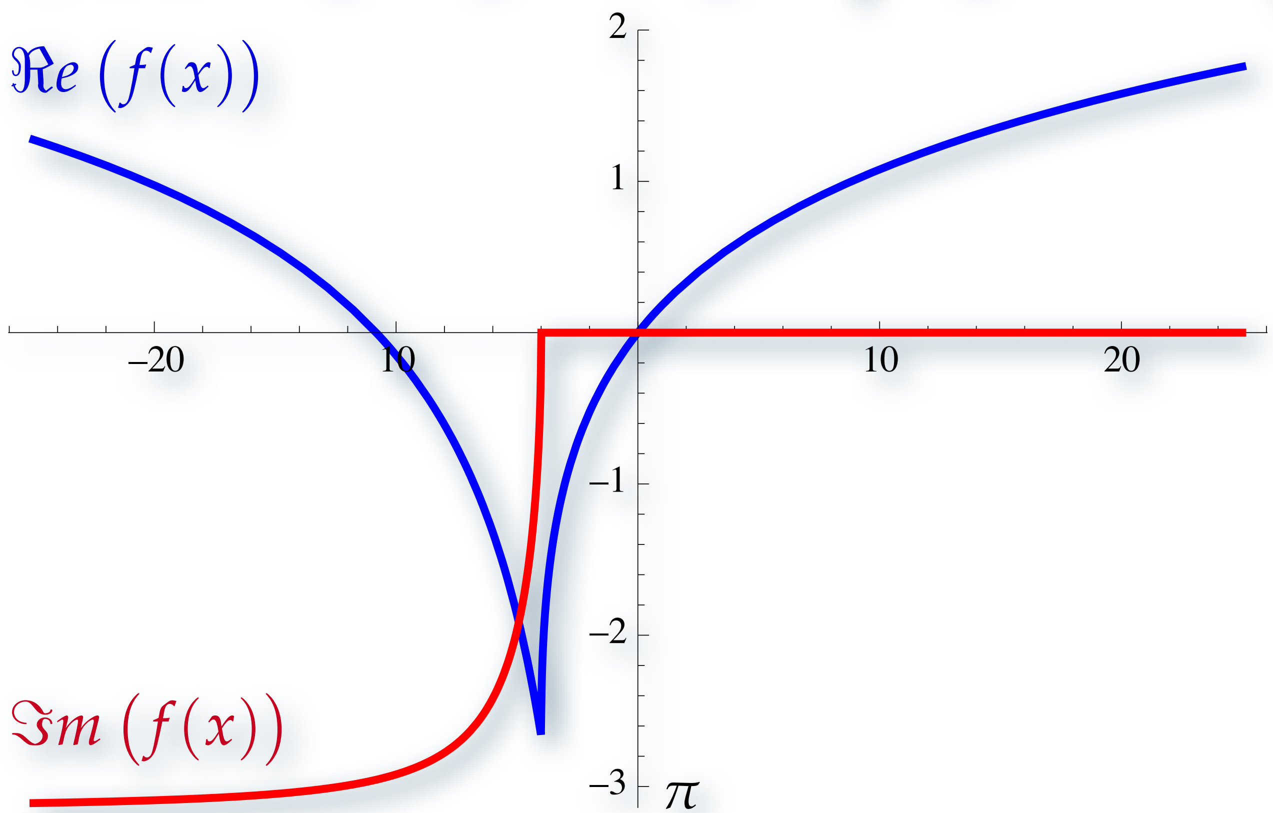
Renormalization

A computation...

- The remaining integral becomes

$$I(q^2) = \frac{g_e^2}{12\pi^2}\left\{ \underbrace{\int_{m_e^2}^{\infty} \frac{d\xi}{\xi}}_{\infty} \underbrace{-6\int_0^1 d\zeta\, \zeta(1-\zeta)\ln\left(1-\frac{q^2}{m_e^2 c^2}\zeta(1-\zeta)\right)}_{f(-q^2/m_e{}^2c^2)} \right\}$$

$$f(x) = \frac{4}{x} - \frac{5}{3} - \frac{2(x-2)}{x}\sqrt{\frac{x+4}{x}}\tan^{-1}\left(\sqrt{\frac{x}{x+4}}\right),$$



$f(x) \sim \ln(x)$ for $|x| \gg 1$

$f(x) \sim x/5$ for $|x| \ll 1$

$\min\left(\Re e(f(x))\right) = -8/3$

$\lim_{x\to-\infty} \Im m\,(f(x)) = -\pi$

$\Im m\,(f(x)) = 0$ for $x \geqslant -4$

Renormalization

A computation...

- The total $(O(g^2)+O(g^4)+\ldots)$ amplitude becomes

$$\mathfrak{M}_{(a)} = \lim_{\mu\to\infty} \mathfrak{M}_{(a)}(\mathrm{q}^2,\mu) + \cdots\,,$$

$$\mathfrak{M}_{(a)}(\mathrm{q}^2,\mu) = -g_R^2(\mu)\,[\overline{u}_3\,\boldsymbol{\gamma}^\mu\,u_1]\left(\frac{\eta_{\mu\nu}}{\mathrm{q}^2}\right)\left\{1+\frac{g_R^2(\mu)}{12\pi^2}\,f\left(\frac{-\mathrm{q}^2}{m_e^2c^2}\right)\right\}[\overline{U}_4\,\boldsymbol{\gamma}^\nu\,U_2] + \cdots\,,$$

$$g_{e,R}(\mu) := g_e\sqrt{1-\frac{g_e^2}{6\pi^2}\ln\left(\frac{\mu}{m_e}\right)},$$

Notice the functional dependence on the 4-momentum exchange

- Amazingly, it is possible to:
  - eliminate the divergent contributions
    - (including all contributions will guarantee appropriate cancellations)
  - compute the "running" coupling constant $g_{e,\mathrm{R}}(\mu)$
    - order by order in $\mathrm{O}(\mathrm{g}^{2n})$, and it tends to converge,
    - ... as a non-trivial function of μ. (The coupling constant ≠ constant.)

Renormalization

The physical meaning

- Conceptual error:
  - The (dimensionless) parameter

$$g_e := \sqrt{4\pi\,\alpha_e} = \frac{|e|}{\sqrt{\epsilon_0 \hbar c}} \qquad \left(= |e|\sqrt{4\pi/\hbar c}, \text{ in Gauss's units}\right)$$

  - used to characterize the strength of the electromagnetic coupling,
  - ... is in classical (field) theory identified with the measured value.
  - But the amplitude—which is what figures in actual, physical measurements of charge—depends on the momentum exchange:

$$\mathfrak{M}_{(a)}(\mathrm{q}^2,\mu) = -g_R^2(\mu)\,[\overline{u}_3\,\boldsymbol{\gamma}^\mu\,u_1]\left(\frac{\eta_{\mu\nu}}{\mathrm{q}^2}\right)\left\{1 + \frac{g_R^2(\mu)}{12\pi^2}\,f\left(\frac{-\mathrm{q}^2}{m_e^2 c^2}\right)\right\}[\overline{U}_4\,\boldsymbol{\gamma}^\nu\,U_2] + \cdots,$$

  - so that the ***measured*** coupling parameter

$$g_{e,R}(\mu) := g_e\sqrt{1 - \frac{g_e^2}{6\pi^2}\ln\left(\frac{\mu}{m_e}\right) + \ldots}$$

  - is definitely ***not*** the one in the original Lagrangian.

Renormalization

The physical meaning

- In addition, there is the reasonable requirement

$$g_{e,R} := \lim_{\mu\to\infty} g_e(\mu)\sqrt{1 - \frac{g_e^2(\mu)}{6\pi^2}\ln\left(\frac{\mu}{m_e}\right) + \dots} \quad < \quad \infty,$$

- …for the "vanishingly small distance" interactions.
- The $\mu \to 0$ limit—at very large distance—better be finite too!
- This can only be arranged when $g_e(\mu)$ is in fact a formally divergent quantity itself.
- Well, in fact, the Lagrangian itself is not an observable.
- The Hamiltonian (energy) ***is***, so that's not quite "***it***."
- It's rather that the measured values of the parameters appearing in the classical Lagrangian are "renormalized" values of the symbols that appear in the classical Lagrangian.

Renormalization

The physical meaning

- Formally: "X"$_{\text{measured}}$ = "X"$_{\text{bare}}$ + "X"$_{\text{renormalization}}$.
  - The "bare" value can be picked at convenience,
  - ...defining what the "renormalization" correction ought to be.
- So, it is convenient to pick

$$g_{e,R}(\mathrm{q}^2) = g_{e,R}(0)\sqrt{1 + \frac{g_{e,R}^2(0)}{12\pi^2}\, f\Big(\frac{-\mathrm{q}^2}{m_e^2 c^2}\Big)},$$

$$\alpha_{e,R}(\mathrm{q}^2) = \alpha_{e,R}(0)\Big\{1 + \frac{\alpha_{e,R}(0)}{3\pi}\, f\Big(\frac{-\mathrm{q}^2}{m_e^2 c^2}\Big) + \ldots\Big\},$$

$$\approx \alpha_{e,R}(0)\Big\{1 + \frac{\alpha_{e,R}(0)}{3\pi}\, \ln\Big(\frac{\mathrm{q}^2}{m_e^2 c^2}\Big) + \ldots\Big\}, \qquad \mathrm{q}^2 \gg m_e^2 c^2,$$

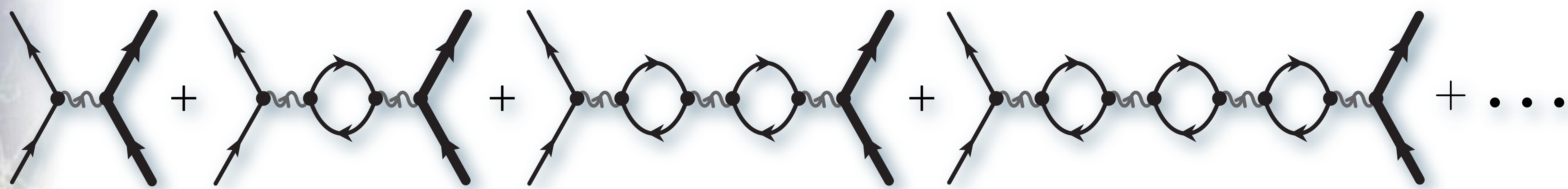
- ...and compare with experiments.
- Indeed: $\alpha_e(0) \approx 1/137$, but $\alpha_e(200\ \text{GeV}) \approx 1/127$.

$1/(2^7+2^3+2^1)$ $1/(2^7-2^2+2^1)$ ⚠☠☣

Renormalization

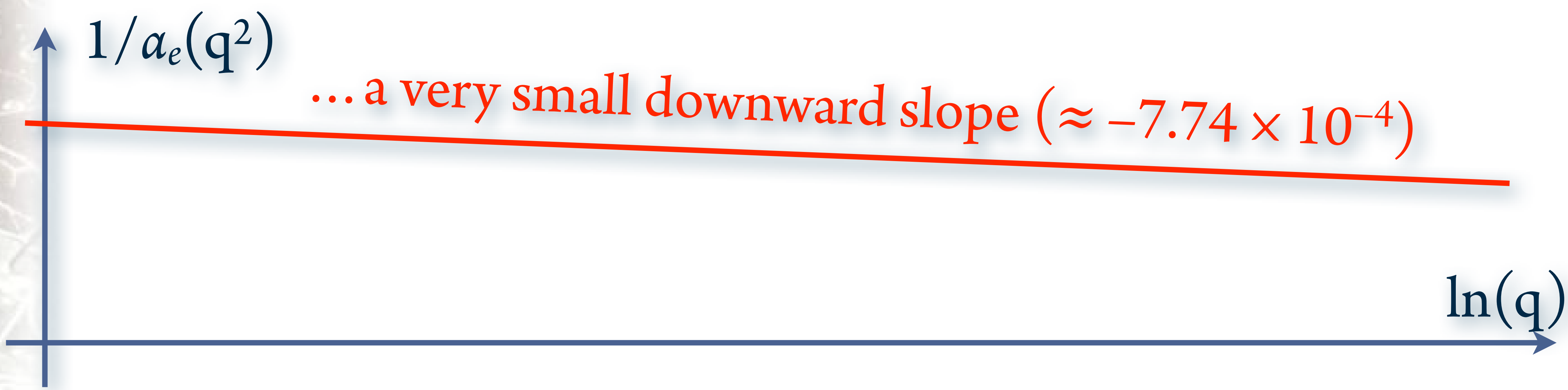
The physical meaning

- The higher-order, but only to "leading log" contributions:



- …defines a geometric series, which sums to:

$$\alpha_{e,R}(|\mathrm{q}^2|) \approx \frac{\alpha_{e,R}(0)}{1 - \frac{\alpha_{e,R}(0)}{3\pi}\ln\left(\frac{|\mathrm{q}^2|}{m_e^2 c^2}\right)}, \quad m_e^2 c^2 \ll \mathrm{q}^2 \ll m_e^2 c^2\, \underbrace{e^{3\pi/2\alpha(0)}}_{\approx 10^{280}}$$



Renormalization

The renormalization "group"

- Computations are done iteratively, order-by-order.
- This defines a sense of "flow":

initial value — intermediate values — real value

$$\Big(\alpha^{(0)}_{e,R}(|q^2|) := \alpha_{e,R}(0)\Big) \mapsto \cdots \mapsto \alpha^{(k)}_{e,R}(|q^2|) \mapsto \alpha^{(k+1)}_{e,R}(|q^2|) \mapsto \cdots \mapsto \alpha^{(\infty)}_{e,R}(|q^2|),$$

$$\mathcal{R}^{(k+1)}_{(k)}$$

- These formal operations form a chain-like algebraic structure.
- Ernst Stückleberg & Andre Petermann, '53 …
  - M. Gell-Mann & F.J. Low '54, (and C. Callan & K. Symanzik '70's)
  - R. Feynman, J. Schwinger, S.-I. Tomonaga ('65 Nobel) & F. Dyson
  - … L.P. Kadanoff '66; K. Wilson '74–75 ('82, Nobel!)
  - + J. Polchinski (1984); M.E. Peskin & D.V. Schröder (1995)
- Renormalization flow…
  - … and fixed points of that flow. ⇒ Quantum stability.

Renormalization

Effective action & partition functional

- Which brings us to the concept of an effective action, and a renormalizable theory.
- Suppose $S[\phi_i] := \int d^4x\, \mathscr{L}(\phi_i, \partial\phi_i, \dots)$
- Then

$$Z[\vartheta] := \int \mathbf{D}[\phi]\; e^{-i(S[\phi]+\int d^4x\,\vartheta\cdot\phi)/\hbar},$$

$$\frac{\delta}{\delta\vartheta^i(x_1)}\frac{\delta}{\delta\vartheta^j(x_2)}Z[\vartheta] = \frac{(-i)^2}{\hbar^2}\int \mathbf{D}[\phi]\;\phi_i(x_1)\,\phi_j(x_2)\; e^{-i(S[\phi]+\int d^4x\,\vartheta\cdot\phi)/\hbar}$$

- Then

$$e^{-i(S_{\text{eff.}}[\varphi]+\int d^4x\,\vartheta\cdot\varphi)/\hbar} := Z[\vartheta] := \int \mathbf{D}[\phi]\; e^{-i(S[\phi]+\int d^4x\,\vartheta\cdot\phi)/\hbar}.$$

a veeery sketchy heuristic of this relation

- …iff the already present parameters become renormalized,
- …in which case the model/theory is renormalizable.

Thanks!

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http://homepage.mac.com/thubsch/