(Fundamental) Physics of Elementary Particles

The Gauge Principle and the $U(1)$ example; Dirac Fermions and Relativistic Electrodynamics

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Fundamental Physics of Elementary Particles

Program

- Non-relativistic $U(1)$ example
  - Non-observable phase & local symmetry
  - Gauge-covariant derivatives
- Electromagnetic fields and Lagrangian
  - Gauge-invariant fields
- Relativistic spinors *A two-part digression*
  - Dirac, Weyl and Majorana spinors
  - Electromagnetic interactions of Dirac spinors
  - Maxwell equations & duality
- Magnetic monopoles, revisited
  - Dirac construction
  - Wu-Yang construction & Dirac quantization

Nonrelativistic *U*(1) Example

Non-Observable Phases & Local Symmetry

- Quantum physics requires assigning to every observable a Hermitian operator (real eigenvalues $\mapsto$ measured values).
- The simplest observable, “does the system/object exist?” is assigned a special Hermitian operator, $\boldsymbol{\rho}$, so that $\mathrm{Tr}[\boldsymbol{\rho}]=1$.
  - In addition, $0 \leq \langle n|\boldsymbol{\rho}|n\rangle \leq 1$ for every $|n\rangle$.
  - All such $\boldsymbol{\rho} = \Sigma_n\, r_n\, |n\rangle\langle n|$, with $0 \leq r_n \leq 1$.
  - For pure states, there exists a $|\Psi\rangle = \Sigma_n\, c_n\, |n\rangle$, such that $\boldsymbol{\rho} = |\Psi\rangle\langle\Psi|$, i.e., $\boldsymbol{\rho}^2 = \boldsymbol{\rho}$, and $\boldsymbol{\rho}$ is a projector.
  - Then $\Psi(\boldsymbol{r}, t) = \langle \boldsymbol{r}|\Psi(t)\rangle$ is the wave-function.
- By construction, $|n\rangle \to e^{i\varphi}|n\rangle$ is a symmetry,
  - since $\boldsymbol{\rho} = \Sigma_n\, r_n\, |n\rangle\langle n| \to \Sigma_n\, r_n\, e^{i\varphi}|n\rangle\langle n|\, e^{-i\varphi} = \Sigma_n\, r_n\, |n\rangle\langle n|$.
  - For pure states, $\Psi(\boldsymbol{r}, t) \to e^{i\varphi}\,\Psi(\boldsymbol{r}, t)$.
  - …Even if $\varphi = \varphi(\boldsymbol{r}, t)$!

Local ~~Gauge~~ symmetry

Nonrelativistic *U*(1) Example

Non-Observable Phases & Local Symmetry

- So, consider the transformation, $|n\rangle \to e^{i\varphi(\boldsymbol{r},t)}|n\rangle$,
  - and for pure states, $\Psi(\boldsymbol{r},t) \to e^{i\varphi(\boldsymbol{r},t)}\Psi(\boldsymbol{r},t)$.
- At every point is space & time, $\varphi(\boldsymbol{r},t) \simeq \varphi(\boldsymbol{r},t)+2\pi$,
  - … $\varphi(\boldsymbol{r},t)$ parametrizes a circle;
  - … $e^{i\varphi(\boldsymbol{r},t)}$ is a unitary number: $(e^{i\varphi(\boldsymbol{r},t)})^\dagger = e^{-i\varphi(\boldsymbol{r},t)} = (e^{i\varphi(\boldsymbol{r},t)})^{-1}$;
  - … $e^{i\varphi(\boldsymbol{r},t)}$ is a unitary 1×1 matrix;
  - … element of the $U(1)$ group;
  - … so $\Psi(\boldsymbol{r},t) \to e^{i\varphi(\boldsymbol{r},t)}\Psi(\boldsymbol{r},t)$ is a $U(1)$ transformation,
    - … a different one at every point in space & time!

If $\varphi(\boldsymbol{r},t) =$ const.
Global symmetry

$\subsetneq$

If $\varphi(\boldsymbol{r},t)$ is arbitrary
Local symmetry

↓ Noether's theorem

↓ Ward-Takashi identities

Quantum Field Theory

Nonrelativistic *U*(1) Example

Gauge-Covariant Derivatives

- So, what's wrong with this picture?
  - Use $\Psi(\boldsymbol{r}, t)$ to describe a particle,
  - …and remember that $\Psi(\boldsymbol{r}, t) \to e^{i\varphi(\boldsymbol{r}, t)}\,\Psi(\boldsymbol{r}, t)$ is unobservable.
- Well, $\Psi(\boldsymbol{r}, t)$ is supposed to satisfy the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\Psi = \Big[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{r},t)\Big]\Psi,$$

$$i\hbar\frac{\partial}{\partial t}\left(e^{i\varphi}\Psi\right) = \Big[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{r},t)\Big]\left(e^{i\varphi}\Psi\right)$$

$$i\hbar\, e^{i\varphi}\left(i\frac{\partial\varphi}{\partial t}\right)\Psi + e^{i\varphi}\, i\hbar\frac{\partial\Psi}{\partial t}$$

$$= -\frac{\hbar^2}{2m}\Big(e^{i\varphi}(i\vec{\nabla}\varphi)^2\Psi + e^{i\varphi}(i\vec{\nabla}^2\varphi)\Psi + 2e^{i\varphi}(i\vec{\nabla}\varphi)\cdot(\vec{\nabla}\Psi)\Big)$$

$$e^{i\varphi}\Big[-\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(\vec{r},t)\Big]\Psi$$

Nonrelativistic *U*(1) Example

Gauge-Covariant Derivatives

- What's left then is a differential equation for $\varphi(\boldsymbol{r}, t)$:

$$\frac{\partial\varphi}{\partial t} = \frac{\hbar}{2m}\left(i(\vec{\nabla}^2\varphi) + 2i(\vec{\nabla}\varphi)\cdot(\vec{\nabla}\ln(\Psi)) - (\vec{\nabla}\varphi)^2\right)$$

- So, far from $\varphi(\boldsymbol{r}, t)$ being a free parameter,
  - … it would have to to satisfy a differential equation,
  - … which moreover depends on $\Psi(\boldsymbol{r}, t)$!
- This is unacceptable.

More to the point …

THIS IS RIDICULOUS!

This just can't be right!

Nonrelativistic *U*(1) Example

Gauge-Covariant Derivatives

- The problem is that

$$\frac{\partial}{\partial t}\left(e^{i\varphi}\Psi\right) \neq e^{i\varphi}\left(\frac{\partial}{\partial t}\Psi\right), \quad \vec{\nabla}\left(e^{i\varphi}\Psi\right) \neq e^{i\varphi}\left(\vec{\nabla}\Psi\right).$$

- …when $\varphi(\mathbf{r}, t) \neq const.$

Duh!

The only thing one ***can*** do.

- So, what is one to do?
- Change the rule of how…
- …derivatives are computed, depending on the symmetry:

$$\left(D_t\Psi\right) \to \left(D_t'\Psi'\right) = D_t'\left(e^{i\varphi}\Psi\right) = e^{i\varphi}\left(D_t\Psi\right),$$

$$\left(\vec{D}\,\Psi\right) \to \left(\vec{D}'\,\Psi'\right) = \vec{D}'\left(e^{i\varphi}\Psi\right) = e^{i\varphi}\left(\vec{D}\,\Psi\right).$$

- Such derivatives would *co-vary* with the local transformation, and so are called ***covariant derivatives***.

Nonrelativistic *U*(1) Example

Gauge-Covariant Derivatives

- With this derivative,

$$
\begin{array}{c}
i\hbar\left(D_t\Psi\right) = -\dfrac{\hbar^2}{2m}\left(\vec{D}^2\,\Psi\right) + V(\vec{r},t)\,\Psi, \\
\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow \\
i\hbar\left(D_t'\Psi'\right) = -\dfrac{\hbar^2}{2m}\left(\vec{D}'^{\,2}\,\Psi'\right) + V(\vec{r},t)\,\Psi', \\
\| \qquad\qquad\qquad \| \qquad\qquad\qquad \backslash\backslash \\
i\hbar\left(D_t' e^{i\varphi}\Psi\right) = -\dfrac{\hbar^2}{2m}\left(\vec{D}'^{\,2}\,e^{i\varphi}\Psi'\right) + V(\vec{r},t)\,e^{i\varphi}\Psi', \\
\| \qquad\qquad\qquad \| \qquad\qquad\qquad \| \\
i\hbar\,\cancel{e^{i\varphi}}\left(D_t\Psi\right) = -\dfrac{\hbar^2}{2m}\cancel{e^{i\varphi}}\left(\vec{D}^2\,\Psi\right) + \cancel{e^{i\varphi}}\,V(\vec{r},t)\,\Psi
\end{array}
$$

- …so the equation transforms with an overall factor of $e^{i\varphi(\boldsymbol{r},\,t)}$,
- …so both $\Psi(\boldsymbol{r}, t)$ and $e^{i\varphi(\boldsymbol{r},\,t)}\,\Psi(\boldsymbol{r}, t)$ satisfy the same equation,
- …and for completely arbitrary and local phase, $\varphi(\boldsymbol{r}, t)$.

Nonrelativistic *U*(1) Example

Gauge-Covariant Derivatives

- So, what are the properties of this new-fangled derivatives?
- By writing $\Psi' = e^{i\varphi}\Psi$, and so $\Psi = e^{-i\varphi}\Psi'$, we have:

$$D_t'\Psi' \stackrel{!}{=} e^{i\varphi} D_t\Psi = e^{i\varphi} D_t e^{-i\varphi}\Psi', \quad \text{or} \quad D_t' = e^{i\varphi} D_t e^{-i\varphi},$$

$$\vec{D}'\Psi' \stackrel{!}{=} e^{i\varphi}\vec{D}\,\Psi = e^{i\varphi}\vec{D}\,e^{-i\varphi}\Psi', \quad \text{or} \quad \vec{D}' = e^{i\varphi}\vec{D}\,e^{-i\varphi}.$$

- This is clearly not true of partial derivatives,
- …nor of so-called total derivatives.
- So, these derivatives must have "correction" terms:

$$\frac{\partial}{\partial t} \to D_t := \frac{\partial}{\partial t} + X, \qquad \vec{\nabla} \to \vec{D} := \vec{\nabla} + \vec{Y}$$

- …for which we work out the local symmetry transformation.

Electromagnetic fields and Lagrangian

Gauge-Covariant Derivatives

- From $(D'_t \cdots) = e^{i\varphi}(D_t e^{-i\varphi} \cdots)$,
- it follows that

$$\Big[\Big(\frac{\partial}{\partial t} + X'\Big)\cdots\Big] = e^{i\varphi}\Big[\Big(\frac{\partial}{\partial t} + X\Big)e^{-i\varphi}\cdots\Big], \qquad X' = X - i\frac{\partial\varphi}{\partial t},$$

- and similarly,

$$[(\vec{\nabla} + \vec{Y}')\cdots] = e^{i\varphi}[(\vec{\nabla} + \vec{Y})e^{-i\varphi}\cdots] \qquad \vec{Y}' = \vec{Y} - i(\vec{\nabla}\varphi).$$

- Thus we obtained the local symmetry transformation rules for the ~~"correction"~~ terms. connexion/connection
- These transformations should look very familiar to anyone who has mastered electromagnetism!
- There, they are referred to as "gauge transformation."

Electromagnetic fields and Lagrangian

Gauge-Covariant Derivatives & Potentials

- Indeed, with a wee bit of judicious rescaling,

$$\Phi := \frac{\hbar}{iq} X, \qquad \vec{A} := \frac{i\hbar}{q}\vec{Y}, \qquad \lambda := \frac{\hbar}{q}\varphi,$$

$$D_t := \frac{\partial}{\partial t} + i\frac{q}{\hbar}\Phi, \qquad \vec{D} := \vec{\nabla} - i\frac{q}{\hbar}\vec{A},$$

- … we recognize the E&M gauge transformations

$$\Phi \to \Phi' = \Phi - \frac{\partial\lambda}{\partial t}, \qquad \vec{A} \to \vec{A}' = \vec{A} + (\vec{\nabla}\lambda),$$

Remember this minus!!

- … augmented by the transformation of the wave-function:

$$\Psi(\vec{r},t) \to \Psi'(\vec{r},t) = e^{iq\lambda(\vec{r},t)/\hbar}\,\Psi(\vec{r},t)$$

- Notice: q is the charge operator, the eigenvalue of which is the charge of the particle represented by Ψ, an eigenfunction.
- So, chargeless particles are not transformed, nor do they require the scalar and vector potentials to interact with.

Electromagnetic fields and Lagrangian

Gauge-Invariant Fields & Maxwell's Equations

- Notice, however, that:

$$\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times (\vec{A} + \vec{\nabla}\lambda) = \vec{\nabla} \times \vec{A}$$

$$\vec{\nabla} \cdot \Phi' + \frac{\partial \vec{A}'}{\partial t} = \vec{\nabla} \cdot \left(\Phi - \frac{\partial \lambda}{\partial t}\right) + \frac{\partial}{\partial t}(\vec{A} + \vec{\nabla}\lambda) = \vec{\nabla} \cdot \Phi + \frac{\partial \vec{A}}{\partial t}$$

- ... are ***invariant*** under the local symmetry transformation.

Maxwell wrote: $\vec{B} := \vec{\nabla} \times \vec{A}$ $\qquad \vec{E} := -\left(\vec{\nabla}\Phi + \frac{\partial \vec{A}}{\partial t}\right)$

- These are indeed the well-known magnetic and electric fields, respectively.
- And, by virtue of these definitions alone:

Maxwell implied: $\vec{\nabla} \cdot \vec{B} = 0,$ as well as $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0.$

50% done.
50% to go.

Electromagnetic fields and Lagrangian

Gauge-Invariant Fields & Maxwell's Equations

- The story so far:
- The Scrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = H_{\rm EM}\,\Psi(\vec{r},t),$$

$$H_{\rm EM} = \frac{1}{2m}\Big(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}(\vec{r},t)\Big)^2 + \Big[V(\vec{r},t) + q\Phi(\vec{r},t)\Big]$$

- ... is ***covariant*** under the local symmetry transformation

$$\Phi \to \Phi' = \Phi - \frac{\partial\lambda}{\partial t}, \qquad \vec{A} \to \vec{A}' = \vec{A} + (\vec{\nabla}\lambda),$$

$$\Psi(\vec{r},t) \to \Psi'(\vec{r},t) = e^{iq\lambda(\vec{r},t)/\hbar}\,\Psi(\vec{r},t),$$

- ... while the following fields are ***invariant***:

$$\vec{B} := \vec{\nabla}\times\vec{A} \qquad \text{and} \qquad \vec{E} := -\Big(\vec{\nabla}\Phi + \frac{\partial\vec{A}}{\partial t}\Big)$$

- ... as are any and all functions of these.

Electromagnetic fields and Lagrangian

Gauge-Invariant Fields & Maxwell's Equations

- The local symmetry transformation $U_\varphi := \exp\{i\varphi(\vec{r},t)\,Q\}$
- …pertains to the phase of wave-functions, φ,
- …which is therefore another "coordinate."
- Q is the charge operator, the eigenvalues of which are the electromagnetic charges of its (wave-)eigenfunctions.
- Willy-nilly, the ***space*** where charged particles "propagate" is:
  - 5-dimensional, of the form $X \times S^1$,
  - where X is the "ordinary" space-time.

1914, Gunnar Nordstrøm
Ch. 7

✎ ***3.1.4*** *Determine the constants c_1, c_2, c_3, c_4, c_5 so that*

$$\int \mathrm{d}t\,\mathrm{d}^3\vec{r}\,\Big\{c_1\big(\epsilon_0\,\vec{E}^2\big)+c_2\big(\tfrac{1}{\mu_0}\,\vec{B}^2\big)+c_3\Big(\sqrt{\tfrac{\epsilon_0}{\mu_0}}\,\vec{E}\cdot\vec{B}\Big)+c_4\,\rho\,\Phi+c_5\,\vec{\jmath}\cdot\vec{A}\Big\} \tag{3.21}$$

is the Hamiltonian action the variation of which by Φ and $\vec{A}$, using the relations (3.14), produces Gauss's and Ampère's law (3.71a).

Relativistic Spinors

Dirac Spinors

- Classical-to-Quantum correspondence

$$\vec{p} \leftrightarrow \vec{p} = \frac{\hbar}{i}\vec{\nabla}, \qquad \text{and} \qquad E \leftrightarrow H = i\hbar\frac{\partial}{\partial t}.$$

- …assigns to the relativistic relation, $E^2 = \boldsymbol{p}^2c^2 + m^2c^4$:

$$\Big[c^2\Big(\frac{\hbar}{i}\vec{\nabla}\Big)^2 + m^2c^4\Big]\Psi(\vec{r},t) = \Big(i\hbar\frac{\partial}{\partial t}\Big)^2\Psi(\vec{r},t),$$

$$\Big[\Box + \Big(\frac{mc}{\hbar}\Big)^2\Big]\Psi(\vec{r},t) = 0,$$

d'Alembertian: $$\Box := \Big[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\Big]$$

- Dirac's motivation:

$$E^2 - m^2c^4 = 0 \qquad \Rightarrow \qquad (E + mc^2)(E - mc^2) = 0,$$

- …in the particle's rest frame; 1st order ODE.

Relativistic Spinors

Dirac Spinors

- So, attempt to factor $\mathrm{p}^2 - m^2 c^2 = 0$

$$0 = (\boldsymbol{\beta}^\mu p_\mu + mc)(\boldsymbol{\gamma}^\nu p_\nu - mc),$$
$$= \boldsymbol{\beta}^\mu \boldsymbol{\gamma}^\nu p_\mu p_\nu + mc(\boldsymbol{\gamma}^\mu - \boldsymbol{\beta}^\mu) p_\mu - m^2 c^2.$$

$$\boldsymbol{\gamma}^\mu \boldsymbol{\gamma}^\nu p_\mu p_\nu = \mathrm{p}^2 \equiv \eta^{\mu\nu} p_\mu p_\nu,$$

$$\{\boldsymbol{\gamma}^\mu, \boldsymbol{\gamma}^\nu\} = 2\eta^{\mu\nu},$$

W.K. Clifford
H. Grassmann
19th century

- …where matrix$[\eta^{\mu\nu}] = \mathrm{diag}(+1,-1,-1,-1)$.
- Pick the second, right-hand factor, so

$$p_\mu \to \frac{\hbar}{i}\partial_\mu \qquad \Rightarrow \qquad [i\hbar\boldsymbol{\gamma}^\mu \partial_\mu - mc]\Psi(\mathrm{x}) = 0,$$

- …is the Dirac equation.

Note the minus sign: $\partial_\mu := \dfrac{\partial}{\partial x^\mu}, \qquad \to (-\tfrac{1}{c}\partial_t, \vec{\nabla})$

Relativistic Spinors

Dirac Spinors

- One oft-used basis of Dirac matrices:

$$\gamma^0 = \begin{bmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & -\mathbb{1} \end{bmatrix}, \qquad \gamma^i = \begin{bmatrix} \mathbb{0} & \sigma^i \\ -\sigma^i & \mathbb{0} \end{bmatrix}, \qquad i = 1, 2, 3.$$

- 4 solutions:

Many other choices & bases...

$$u^\uparrow \propto \begin{bmatrix} 1 \\ 0 \\ \frac{p_z c}{E+mc^2} \\ \frac{(p_x+ip_y)c}{E+mc^2} \end{bmatrix}, \qquad u^\downarrow \propto \begin{bmatrix} 0 \\ 1 \\ \frac{(p_x-ip_y)c}{E+mc^2} \\ \frac{p_z c}{E+mc^2} \end{bmatrix},$$

$$v^\downarrow \propto \begin{bmatrix} \mathbb{0} & \mathbb{1} \\ \mathbb{1} & \mathbb{0} \end{bmatrix} u^\uparrow \propto \begin{bmatrix} \frac{p_z c}{E+mc^2} \\ \frac{(p_x+ip_y)c}{E+mc^2} \\ 1 \\ 0 \end{bmatrix}, \qquad v^\uparrow \propto \begin{bmatrix} \mathbb{0} & \mathbb{1} \\ \mathbb{1} & \mathbb{0} \end{bmatrix} u^\downarrow \propto \begin{bmatrix} \frac{(p_x-ip_y)c}{E+mc^2} \\ \frac{p_z c}{E+mc^2} \\ 0 \\ 1 \end{bmatrix},$$

- where $E = +\sqrt{\vec{p}^2 c^2 + m^2 c^4}$

$$\Psi(\mathrm{x}) = \sum_{s=\uparrow,\downarrow} \left[N_u \, e^{-(i/\hbar)\mathrm{x}\cdot\mathrm{p}} \, u^s(\mathrm{p}) + N_v \, e^{-(i/\hbar)\mathrm{x}\cdot\mathrm{p}} \, v^s(\mathrm{p}) \right]$$

Relativistic Spinors

Dirac Matrices & the Lorentz Group

- Dirac matrices satisfy

$$[\boldsymbol{\gamma}^{\mu\nu}, \boldsymbol{\gamma}^{\rho\sigma}] = \eta^{\mu\rho}\boldsymbol{\gamma}^{\nu\sigma} - \eta^{\mu\sigma}\boldsymbol{\gamma}^{\nu\rho} + \eta^{\nu\sigma}\boldsymbol{\gamma}^{\mu\rho} - \eta^{\nu\rho}\boldsymbol{\gamma}^{\mu\sigma}.$$

- Or, with the definitions $J_j := \frac{1}{2i}\varepsilon_{jkl}\boldsymbol{\gamma}^{kl} \quad K_j := i\boldsymbol{\gamma}^{0j}$

$$[J_j, J_k] = i\varepsilon_{jk}{}^{m} J_m, \qquad [J_j, K_k] = i\varepsilon_{jk}{}^{m} K_m, \qquad [K_j, K_k] = -i\varepsilon_{jk}{}^{m} J_m.$$

- So, the elements

$$\exp\{-i(\varphi^j J_j + \beta^j K_j)\} = \exp\{\beta_j \boldsymbol{\gamma}^{0j} - \varepsilon_{jkm}\varphi^j \boldsymbol{\gamma}^{km}\} = \exp\{\lambda_{\mu\nu}\boldsymbol{\gamma}^{\mu\nu}\},$$

- …form a group, equivalent to $SO(1,3)$, except that…
- …it is single-valued on spin-½ Dirac spinors.
- A x^1-Lorentz-boost:

$$\Psi(\mathrm{x}) \to \left[\sqrt{\tfrac{1}{2}(\gamma+1)}\,\mathbb{1} - \sqrt{\tfrac{1}{2}(\gamma-1)}\,\boldsymbol{\gamma}^{01}\right]\Psi(\mathrm{x})$$

Relativistic Spinors

Dirac Matrices & the Lorentz Group

- Note: $\Psi^{\dagger}\Psi$ is not Lorentz-invariant, but $\Psi^{\dagger}\boldsymbol{\gamma}^{0}\Psi$ is.
- Define the Dirac conjugate: $\overline{\Psi} := \Psi^{\dagger}\boldsymbol{\gamma}^{0}$

Expression	Lorentz representation	# of Independent Components
$\overline{\Psi}\,\Psi$	$=$ scalar,	1
$\overline{\Psi}\,\boldsymbol{\gamma}^{\mu}\,\Psi$	$=$ 4-vector,	4
$\overline{\Psi}\,\boldsymbol{\gamma}^{\mu\nu}\,\Psi$	$=$ antisymmetric rank-2 tensor,	6
$\overline{\Psi}\,\boldsymbol{\gamma}^{\mu}\widehat{\boldsymbol{\gamma}}\,\Psi$	$=$ axial (*i.e.*, pseudo-) 4-vector,	4
$\overline{\Psi}\,\widehat{\boldsymbol{\gamma}}\,\Psi$	$=$ pseudo-scalar,	1

- ...is a complete set of Lorentz-representations constructed from the spin- Dirac spinor representation.
- Notice, Ψ is a "square-root" of the vector.
- In the same sense $Spin(1,3)$ is the double-cover of $SO(1,3)$.

Relativistic Spinors

Back to Dirac Spinors

- The solutions $u^{\uparrow}$, $u^{\downarrow}$, $v^{\uparrow} \propto \boldsymbol{\gamma}^1 u^{\downarrow}$ and $v^{\downarrow} \propto \boldsymbol{\gamma}^1 u^{\uparrow}$:

$$\sum_{s=\uparrow,\downarrow} u^{s,A}\,\overline{u^s}_B = (\gamma^\mu)^A{}_B\, p_\mu + mc\,\delta^A_B,$$

$$\sum_{s=\uparrow,\downarrow} v^{s,A}\,\overline{v^s}_B = (\gamma^\mu)^A{}_B\, p_\mu - mc\,\delta^A_B.$$

- Note: $\boldsymbol{\gamma}_\pm := \frac{1}{2}[\mathbb{1} \pm \widehat{\boldsymbol{\gamma}}]$, are projection matrices.

$$\Psi_\pm := \boldsymbol{\gamma}_\pm \Psi,$$

Weyl spinors

- Notice, however:

$$\Psi_+ + \Psi_- = \Psi, \quad \boldsymbol{\gamma}_\pm \Psi_\pm = \Psi_\pm, \quad \boldsymbol{\gamma}_\pm \Psi_\mp = 0.$$

$$[\boldsymbol{\gamma}_\pm, \boldsymbol{\gamma}^\mu] \neq 0: \qquad [\mathbb{1}, \boldsymbol{\gamma}^\mu] = 0 = \{\widehat{\boldsymbol{\gamma}}, \boldsymbol{\gamma}^\mu\} \quad \Rightarrow \quad \boldsymbol{\gamma}_\pm \boldsymbol{\gamma}^\mu = \boldsymbol{\gamma}^\mu \boldsymbol{\gamma}_\mp.$$

$$\begin{aligned} \boldsymbol{\gamma}_\pm [i\hbar \boldsymbol{\gamma}^\mu \partial_\mu - mc\mathbb{1}]\Psi &= [i\hbar \boldsymbol{\gamma}_\pm \boldsymbol{\gamma}^\mu \partial_\mu - mc\boldsymbol{\gamma}_\pm \mathbb{1}]\Psi \\ &= i\hbar \boldsymbol{\gamma}^\mu (\partial_\mu \Psi_\mp) - mc\Psi_\pm, \end{aligned}$$

Relativistic Spinors

Back to Dirac Spinors

- So, splitting the 4-component Dirac spinor into two 2-component Weyl spinors is ***dynamically*** possible if:

$$\boldsymbol{\gamma}^{\mu}\partial_{\mu}\Psi_{\pm} = 0 \qquad\qquad m\Psi_{\pm} = 0.$$

- Weyl noticed this in 1929, immediately after Dirac announced the Dirac equation.
- Ironically, although Pauli knew that $m_{\nu} \approx 0$, he refused to use Weyl's equation on account of it allowing parity-violation:
  - $P(\Psi_{+}) = \Psi_{-}$ as well as $P(\Psi_{-}) = \Psi_{+}$,
  - but Weyl's equations decouple Ψ_{+} from Ψ_{-},
  - …so they can be treated independently.
- Dirac's Lagrangian:

$$\mathscr{L}_D = -\overline{\Psi}(\mathrm{x})\left[c\not{p} + mc^2\mathbb{1}\right]\Psi(\mathrm{x}) = \overline{\Psi}(\mathrm{x})\left[i\hbar c\boldsymbol{\gamma}^{\mu}\partial_{\mu} - mc^2\mathbb{1}\right]\Psi(\mathrm{x})$$

Electromagnetic fields and Lagrangian

Maxwell Equations & Duality

- Maxwell's equations

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{4\pi\epsilon_0}4\pi\,\rho_e,$$

Gauss

$$\vec{\nabla}\cdot(c\vec{B}) = \frac{\mu_0}{4\pi}4\pi\,\rho_m,$$

$$\vec{\nabla}\times(c\vec{B}) - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{1}{4\pi\epsilon_0}\frac{4\pi}{c}\vec{j}_e,$$

Ampère

Faraday

$$-\vec{\nabla}\times\vec{E} - \frac{1}{c}\frac{\partial(c\vec{B})}{\partial t} = \frac{\mu_0}{4\pi}\frac{4\pi}{c}\vec{j}_m,$$

- can be re-cast into relativistic, 4-vector/tensor notation.

$$A_\mu := (\Phi, -c\vec{A}), \qquad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu,$$

- Linear action by Lorentz transformations:

$$y^\mu = \mathrm{L}^\mu{}_\nu x^\nu \quad\Rightarrow\quad F_{\mu\nu}(\mathrm{y}) = \mathrm{L}_\mu{}^\rho\, F_{\rho\sigma}(\mathrm{x})\,\mathrm{L}^\sigma{}_\nu.$$

Let $\vec{E} = \hat{e}^2 E_2$ and $\vec{B} = 0 \Rightarrow \widetilde{E}_2 = \gamma E_2,$ but also $\widetilde{B}_3 = \gamma\frac{v_1}{c^2}E_2.$

Electromagnetic fields and Lagrangian

Maxwell Equations & Duality

- Gauss-Ampère equations:

$$\partial_\mu F^{\mu\nu} = \frac{1}{4\pi\epsilon_0}\frac{4\pi}{c} j_e^\nu$$

- Gauss-Faraday equations:

$$\tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\nu\rho} = \frac{\mu_0}{4\pi}\frac{4\pi}{c} j_m^\sigma,$$

- Direct substitutions:

$$\tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} = \vec{E}^2 - c^2\vec{B}^2, \qquad \text{and} \qquad \tfrac{1}{4}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = -c\vec{E}\cdot\vec{B},$$

$$\mathscr{L}_{EM} = -\frac{4\pi\epsilon_0}{4}F_{\mu\nu}F^{\mu\nu} \qquad \mathscr{L}_{\vartheta,EM} = \vartheta\,\frac{4\pi\epsilon_0}{4}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma},$$

- Note:

$$\tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\left(\partial_\nu A_\rho - \partial_\rho A_\nu\right) = \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\nu A_\rho - \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\rho A_\nu,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad\Longleftrightarrow\quad 0 = \frac{\mu_0}{4\pi}\frac{4\pi}{c} j_m^\sigma.$$

Electromagnetic fields and Lagrangian

Maxwell Equations & Duality

- Duality:

$$\varpi_{EM} \,:\, F^{\mu\nu} \longleftrightarrow (*F)^{\mu\nu} = [\tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}]$$

$$\varpi_{EM}(\vartheta) : \begin{bmatrix} \vec{E} \\ c\vec{B} \end{bmatrix} \to \begin{bmatrix} \vec{E}' \\ c\vec{B}' \end{bmatrix} = \begin{bmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{bmatrix} \begin{bmatrix} \vec{E} \\ c\vec{B} \end{bmatrix}$$

- Thus: “there are no magnetic monopoles” = “there is a choice of ϑ, such that $\rho_m = 0 = \vec{\jmath}_m$, $\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$ $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
- whereby the Gauss-Ampère equations (using the Lorenz gauge: $\partial_\mu A^\mu = 0$) imply

$$\Box A^\mu = \frac{1}{4\pi\epsilon_0}\frac{4\pi}{c} j_e^\mu,$$

- and A^μ represents a massless field of which j_e^μ is the source.
- …of which the photon is the quantum.

Electromagnetic fields and Lagrangian

Electrodynamics of a Dirac Spinor

- Combining the EM Lagrangian, the Dirac Lagrangian, and the coupling enforced by the local (gauge) transformation

$$\Psi(\vec{r},t) \to \Psi'(\vec{r},t) = e^{iq\lambda(\vec{r},t)/\hbar}\,\Psi(\vec{r},t)$$

$$\partial_\mu \to D_\mu := \partial_\mu + \frac{i}{\hbar c} A_\mu\, Q,$$

$$\mathscr{L}_{QED} = \overline{\Psi}(\mathrm{x})\,[i\hbar c \not{D} - mc^2]\,\Psi(\mathrm{x}) - \frac{4\pi\epsilon_0}{4} F_{\mu\nu}F^{\mu\nu},$$

$$= \overline{\Psi}(\mathrm{x}) \left[\boldsymbol{\gamma}^\mu (\hbar c\, i\partial_\mu - q_\Psi A_\mu) - mc^2\right] \Psi(\mathrm{x})$$

$$- \frac{4\pi\epsilon_0}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)\eta^{\mu\rho}\eta^{\nu\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho).$$

- Now, just include a separate copy of Ψ for every separate spin-½ fermion, with its appropriate charge q_Ψ.
- … and the rest is work. Hard work.

Magnetic Monopoles, Revisited

Dirac Construction

- Magnetic monopoles, nevertheless…
- P.A.M. Dirac, 1931:
  - Consider an ε-narrow, long solenoid,
  - …with the North pole at the origin,
  - …the South pole at $1/\varepsilon$ below the origin,
  - …in the limit when $\varepsilon \longrightarrow 0$.
- What you see (= what you get) is
  - a magnetic monopole (North) at the origin,
  - with a perfectly spherically symmetric magnetic field.
- Dirac showed that this $\boldsymbol{B} \propto q_m\, \boldsymbol{r}/r^3$ must be ill-defined
  - …along a branch-cut, extending "from the origin to infinity."
  - = "Dirac string."

Magnetic Monopoles, Revisited

Dirac Construction

- For this magnetic monopole, Dirac derived that

$$q_e\, q_m = 2\pi\hbar\, n, \qquad n \in \mathbb{Z}.$$

- This implies:

$$\frac{\alpha_m}{\alpha_e} \approx 4\,690\, n^2$$

- … and since $\alpha_e \sim 1/137$, $\alpha_m \sim 34.25\, n^2 \gg 1$!!!
- In fact, this will turn out to be a general feature:
  - if "electric"-charged particles interact perturbatively weakly,
  - "magnetic"-charged particles interact non-perturbatively strongly!
- Dirac seems to have pulled something really "funny" here!
- Recall: $\rho_m = 0 = \vec{\jmath}_m, \quad \Longleftrightarrow \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
- So, like … *What gives??*

Magnetic Monopoles, Revisited

Wu-Yang Construction & Dirac Quantization

- T.T. Wu & C.N. Yang, 1975 (months after the "November revolution of 1974, the J/ψ, quark-model and all that):

$$\vec{A}_N = \frac{q_m}{4\pi}\frac{x\,\hat{e}_y - y\,\hat{e}_x}{r(z+r)}, \qquad \vec{A}_S = \frac{q_m}{4\pi}\frac{x\,\hat{e}_y - y\,\hat{e}_x}{r(z-r)},$$

$$= -\frac{q_m}{4\pi}\frac{\cos(\theta)-1}{r\sin(\theta)}\,\hat{e}_\phi, \qquad = -\frac{q_m}{4\pi}\frac{\cos(\theta)+1}{r\sin(\theta)}\,\hat{e}_\phi,$$

$$\vec{B}_N := \vec{\nabla}\times\vec{A}_N = \frac{q_m}{4\pi}\frac{\vec{r}}{r^3}, \quad \text{and} \quad \vec{B}_S := \vec{\nabla}\times\vec{A}_S = \frac{q_m}{4\pi}\frac{\vec{r}}{r^3}.$$

(except where $x = 0 = y$ and $z \leqslant 0$) (except where $x = 0 = y$ and $z \geqslant 0$)

- … and

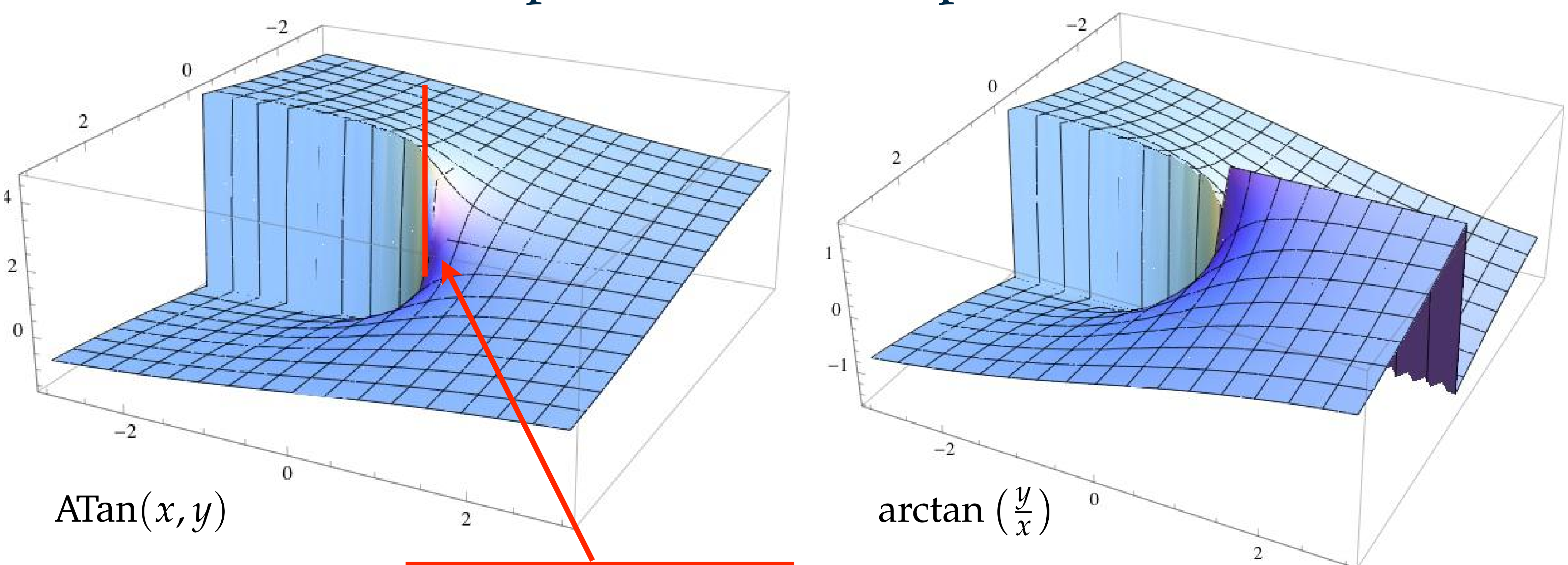
$$\vec{A}_N \to \vec{A}_S = \vec{A}_N + \vec{\nabla}\lambda_{NS}, \quad \lambda_{NS}(\mathbf{x}) = -2\frac{q_m}{4\pi}\,\mathrm{ATan}(x,y)$$

- … is a local (gauge) symmetry transformation.

Magnetic Monopoles, Revisited

Wu-Yang Construction & Dirac Quantization

- Lessons:
  - Note that the gauge parameter $\lambda_{NS} \propto \mathrm{ATan}(x, y)$ seems undefined along the (x, z)-plane, where $y = 0$.
  - However, its limiting values there do define a continuous and smooth function, with periodic values of period 2π:



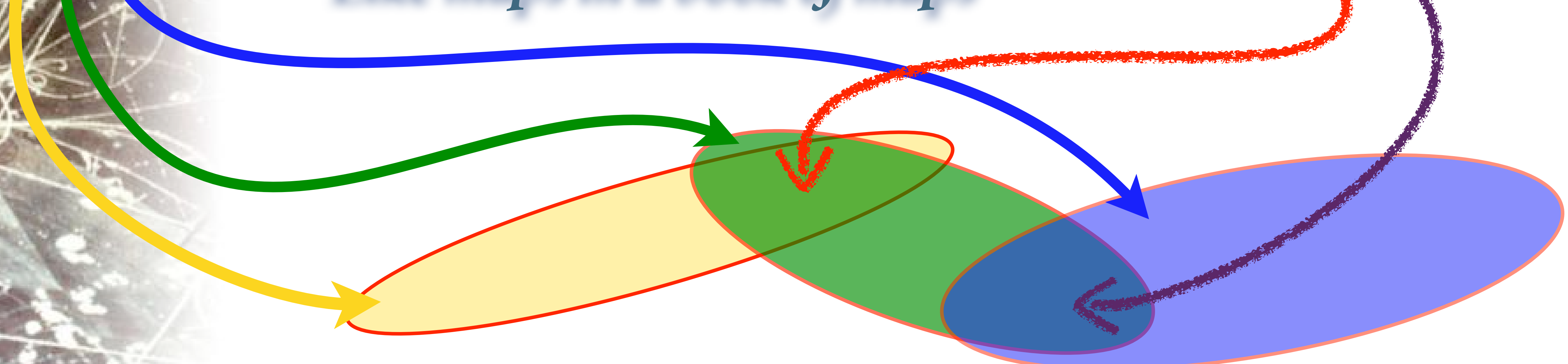
  - …undefined only along the z-axis.

Magnetic Monopoles, Revisited

Wu-Yang Construction & Lessons

- In a local (gauge) symmetry model,
  - The observable physical quantities (electric & magnetic fields) need to be well-defined in all physically accessible space.
  - Auxiliary quantities (scalar and vector potentials) may be defined only "piecemeal"—and need not be globally well-defined.
  - Gauge parameters ("patching" functions such as λ_{NS}) need be defined only where the "piecemeal" auxiliary quantities' defining regions overlap, so as to "stitch" them together.

Like maps in a book of maps



Thanks!

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