(Fundamental) Physics of Elementary Particles

Finite symmetries & the CPT-theorem; Isospin and the SU(2) group and its representations

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Fundamental Physics of Elementary Particles

Program

- Finite Symmetries
  - Parity (Space Reflection)
  - Charge Conjugation
  - Time Reversal
  - CPT Theorem
- Isospin
  - Introduction
  - $SU(2)$ formalism
  - Applications
- Quark Bound States and $SU_f(3)$
  - Light Meson Masses
  - Light Baryon Masses and Magnetic Moments

Finite Symmetries

Parity

- Cartesian space coordinate reflection
  - Reflecting one coordinate ≃ plane mirror reflection
  - Reflecting all space coordinates (in odd-dim'l space)
  - Squares to $\mathbb{1}$ (the "do nothing" identity operator),
    - eigenvalue = +1: symmetric eigenstate (e.g., $\cos(x)$),
    - eigenvalue = –1: antisymmetric eigenstate (e.g., $\sin(x)$).
  - This is definitely *not* a symmetry in the macroscopic world
- It was believed to be a symmetry of fundamental physics
  - For example, Pauli refused to use the Weyl equation for neutrinos
  - …although their mass was known to be possibly zero…
  - …because parity is not a symmetry of the Weyl equation
- Parity of a composite system equals
  - $(\text{parity}_1)\cdot(\text{parity}_2)\cdot(-1)^{\ell}$, where ℓ = orbital angular momentum.

Finite Symmetries

Parity

- T.D. Lee & C.N. Yang (1956) studied the decays:

$$\text{“}\theta^{+}\text{”} \to \pi^+ + \pi^0, \qquad \text{“}\tau^{+}\text{”} \begin{array}{l} \nearrow \pi^+ + \pi^0 + \pi^0 \\ \searrow \pi^+ + \pi^+ + \pi^- \end{array}$$

- Spin(θ^+)=0=Spin(τ^+), Parity(θ^+)=+1, while Parity(τ^+)=–1.
- Lee & Yang: they are the same particle, but weak interactions violate parity.
- Proposed several tests of P-violation in weak interactions.
- C.S. Wu (+E. Ambler, R.W. Hayward, D.D. Hoppes and R.P. Hudson) tested

$$^{60}_{27}\mathrm{Co} \to {}^{60}_{28}\mathrm{Ni} + e^- + \bar{\nu}_e,$$

- …and found most electrons to emerge in correlation with the spin of Cobalt-60:

$$\langle \Psi | \vec{p}_e \cdot \vec{S} | \Psi \rangle \neq 0.$$

Finite Symmetries

Parity

- Now,
$$\langle\Psi|\vec{p}_e\cdot\vec{S}|\Psi\rangle \stackrel{P}{=} \langle\Psi'|\vec{p}_e'\cdot\vec{S}'|\Psi'\rangle = -\langle\Psi'|\vec{p}_e\cdot\vec{S}|\Psi'\rangle.$$
- In turn, if $[H\,, P] = 0$, *i.e.*, P is a symmetry, then
- either $P|\Psi\rangle = |\Psi'\rangle = c\,|\Psi\rangle$, so $P\langle\Psi| = \langle\Psi'| = c^*\,\langle\Psi|$, so
$$\langle\Psi|\vec{p}_e\cdot\vec{S}|\Psi\rangle = 0, \quad \text{NOT what Wu saw,}$$
- or $P|\Psi\rangle = |\Psi'\rangle \neq c\,|\Psi\rangle$, so that $|\Psi'\rangle$ and $|\Psi\rangle$ are degenerate,
- …which does not agree with otherwise good nuclear models.
- It follows that $[H\,, P] \neq 0$, *i.e.*, P is not a symmetry.
- Once uncovered, P-violation was confirmed elsewhere too.
- Right-handed neutrinos differ from the left-handed ones
- No more than 10^{-10} observed neutrinos may be right-handed.

Finite Symmetries

Parity

- For example,

$$\pi^- \to \mu^- + \bar{\nu}_\mu$$

- In the pion's rest-frame, the muon and the antineutrino
  - move in opposite directions, and
  - have $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ spins.
- While the antineutrinos are largely unobservable,
- … at most 10^{-10} muons may emerge left-handed.
- Thus at most 10^{-10} of the antineutrinos are left-handed,
- … and at most 10^{-10} of the neutrinos are right-handed.
- This implies ***maximal*** parity-violation in weak interactions.

Finite Symmetries

Charge Conjugation

- An *anti-linear* operation:
- $$\mathsf{C}\big(c_1\,|\Psi_1\rangle + c_2\,|\Psi_2\rangle\big) = c_1^*\,\mathsf{C}\big(|\Psi_1\rangle\big) + c_2^*\,\mathsf{C}\big(|\Psi_2\rangle\big).$$
- This forces the operation also to be anti-unitary (unitary preceded by complex conjugation).
- It is identifiable with complex/Hermitian/Dirac conjugation.
- Eigenstates must have no electric charge if C is a symmetry of electromagnetism.
- Eigenstates must have no color "charge" if C is a symmetry of chromodynamics (strong interactions).
- If C is a symmetry of the "free" Hamiltonian, $[H\,,\mathsf{C}] = 0$ which is true if H is Hermitian, then $|\Psi\rangle$ and $\mathsf{C}(|\Psi\rangle)$ are degenerate.
- Indeed, e.g., e^- and e^+ have the same mass.

Finite Symmetries

Charge Conjugation

- A bound state of a particle and its antiparticle
- ... is an eigenstate of C, with the eigenvalue $(-1)^{\ell+s}$,
- ℓ is the orbital angular momentum, s the composite spin.
- In the electromagnetic decay of the pion,

$$\pi^0 \to \gamma + \gamma$$

- ... there can only be an even number of photons.
- But, C is violated by weak interactions:

$$\mathsf{C}(\pi^- \to \mu_R^- + \bar{\nu}_\mu) \;=\; \pi^+ \to \mu_R^+ + \nu_\mu,$$

- ... does not occur... more than at most 10^{-10} part of the time.
- C is thus also ***maximally*** violated by weak interactions.

Finite Symmetries

Time Reversal

- Also an anti-linear and anti-unitary operation.
- Direct verification of T-violation is hard…
- No physically meaningful eigenstates.
- In processes, T-violation could be checked by checking the "principle of detailed balance":
- Compare

$$\Lambda^0 \to p^+ + \pi^- \qquad vs. \qquad p^+ + \pi^- \to \Lambda^0$$

- But, the latter (fusion) is swamped by other results, due to strong and electromagnetic interactions.
- The only (solely weak interaction) direct tests of T-violation then involve neutrinos… and those are extreeemely hard.

Finite Symmetries

CPT Theorem

- The successive application of the C,- P- and T-operations is a symmetry of all local Lorentz-invariant theories.
- A simple-looking argument begins with

$$\begin{aligned} CPT(e^{\pm i(\vec{k}\cdot\vec{r}-\omega t)}) &= CP(e^{\pm i(\vec{k}\cdot\vec{r}+\omega t)}) = C(e^{\pm i(\vec{k}\cdot(-\vec{r})+\omega t)}) = (e^{\mp i(\vec{k}\cdot(-\vec{r})+\omega t)}), \\ &= e^{\pm i(\vec{k}\cdot\vec{r}-\omega t)}, \end{aligned}$$

- …and then uses that plane-waves form a complete set of Lorentz-invariant functions.
- Thus, all spacetime-dependent Lorentz-invariant observables may be expressed in terms of plane-waves, and so must also be CPT-invariant.
- The difficulty lies in proving no loss of generality.

Finite Symmetries

CP-Violation

- Direct T-violation is very hard, but CP-violation is not *that* hard.
- J. Cronin and V. Fitch (1964) surprised everyone with their experimental proof of CP-violation…
- …related to a 10-year old observation by M. Gell-Mann and A. Pais: K^0 cannot be its anti-particle:

$$\pi^-_{(0)} + p^+_{(0)} \to \Lambda^0_{(-1)} + K^0_{(+1)}, \qquad \text{and} \qquad \pi^+_{(0)} + p^+_{(0)} \to p^+_{(0)} + \overline{K}^0_{(-1)} + K^+_{(+1)}.$$

- Kaons are also *pseudo*-scalars:

$$\mathsf{CP}|K^0\rangle = -\mathsf{C}|K^0\rangle = -|\overline{K}^0\rangle, \qquad \mathsf{CP}|\overline{K}^0\rangle = -\mathsf{C}|\overline{K}^0\rangle = -|K^0\rangle,$$

- so CP-eigenstates are:

$$|K^0_+\rangle := \tfrac{1}{\sqrt{2}}\big(|K^0\rangle - |\overline{K}^0\rangle\big), \qquad \text{and} \qquad |K^0_-\rangle := \tfrac{1}{\sqrt{2}}\big(|K^0\rangle + |\overline{K}^0\rangle\big)$$

Finite Symmetries

CP-Violation

- Since

$$CP|K^0_\pm\rangle = (\pm 1)|K^0_\pm\rangle$$

- $K^0{}_+$ decays into two pions, $K^0{}_-$ into three.

$$\left(\Gamma_{K^0_+} \propto \sqrt{1-(2m_{\pi^0}/m_{K^0})^2}\right) > \left(\Gamma_{K^0_-} \propto \sqrt{1-(3m_{\pi^0}/m_{K^0})^2}\right)$$

$$\tau_{K^0_+} < \tau_{K^0_-}$$

- Indeed,

$$\tau_{K^0_+} = 0.8958\times 10^{-10}\,\text{s}, \qquad \text{while} \qquad \tau_{K^0_-} = 5.114\times 10^{-8}\,\text{s}.$$

- So, $K^0{}_-$ lives some 570 times longer than $K^0{}_+$.

Finite Symmetries

CP-Violation

- Created in pairs by strong interactions, their 50-50% distribution soon changes:

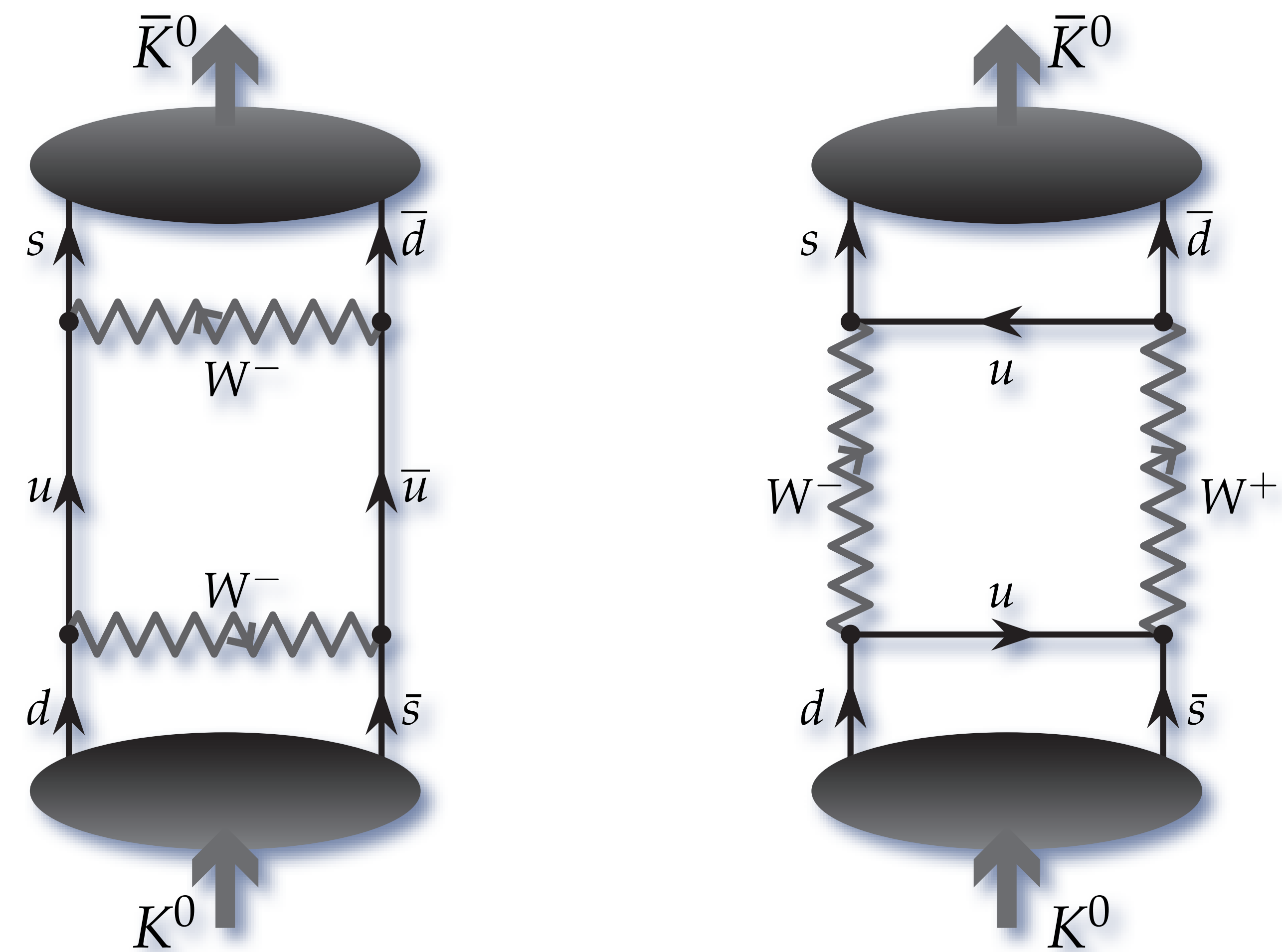
$$\frac{N(K^0_+)}{N(K^0_-)} = \frac{e^{-t/\tau^+}}{e^{-t/\tau^-}} = \exp\left\{-\frac{t}{\tau^+}+\frac{t}{\tau^-}\right\} \approx \exp\left\{-569\,\frac{t}{\tau^-}\right\}$$

- …is ~1.47×10^{-5} after even just 1 ns.
- Cronin & Fitch found 2-pion decays even after long times,
- …proving that the distinct *CP*-eigenstates transmogrify one into another.
- Also,

$$K^0_- \begin{cases} \to \pi^+ + e^- + \bar{\nu}_e \\ \to \pi^- + e^+ + \nu_e \end{cases}$$

CP

favored by δ~3.3×10^{-3}



Isospin

SU(2) Formalism

- W. Heisenberg (1932): differences between p^+ and n^0 are irrelevant to the strong interactions.
- p^+ and n^0 are akin to "spin-up" and "spin-down" *nucleons*.
- E. Wigner (1937) called this *isospin*:

$$\vec{I}: \quad \left[\, I_j \,,\, I_k \,\right] = i\varepsilon_{jk}{}^{m}\, I_m$$

$$|I, I_3\rangle : \quad I^2|I, I_3\rangle = I(I{+}1)|I, I_3\rangle, \quad I_3|I, I_3\rangle = I_3|I, I_3\rangle, \quad I = \max(|I_3|),$$

$$I_\pm|I, I_3\rangle = \sqrt{I(I{+}1) - I_3(I_3{\pm}1)}\,|I, I_3{\pm}1\rangle, \qquad \triangle I_3 \in \mathbb{Z}.$$

- Like angular momentum, isospin can only be integral or half-integral…
- …and may well be even added to angular momentum.

$$|p^+\rangle = |\tfrac{1}{2}, +\tfrac{1}{2}\rangle, \qquad |n^0\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle.$$

Isospin

Applications

- Once a few other hardons' isospin states are identified,

$$|\pi^+\rangle = |1,+1\rangle, \qquad |\pi^0\rangle = |1,0\rangle, \qquad |\pi^-\rangle = |1,-1\rangle,$$

$$|\Delta^{++}\rangle = |\tfrac{3}{2},+\tfrac{3}{2}\rangle, \quad |\Delta^{+}\rangle = |\tfrac{3}{2},+\tfrac{1}{2}\rangle, \quad |\Delta^{0}\rangle = |\tfrac{3}{2},-\tfrac{1}{2}\rangle, \quad |\Delta^{-}\rangle = |\tfrac{3}{2},-\tfrac{3}{2}\rangle,$$

- we can compute!

$$\begin{aligned}
a:&\quad p^+ + p^+ \to d + \pi^+, &&\leftrightarrow\quad |\tfrac{1}{2},+\tfrac{1}{2}\rangle|\tfrac{1}{2},+\tfrac{1}{2}\rangle \to |0,0\rangle|1,+1\rangle = |1,+1\rangle,\\
b:&\quad p^+ + n^0 \to d + \pi^0, &&\leftrightarrow\quad |\tfrac{1}{2},+\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle \to |0,0\rangle|1,0\rangle \quad = |1,0\rangle,\\
c:&\quad n^0 + n^0 \to d + \pi^0, &&\leftrightarrow\quad |\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle \to |0,0\rangle|1,-1\rangle = |1,-1\rangle.
\end{aligned}$$

$$\begin{aligned}
\mathfrak{M}_a \propto \langle d,\pi^+|p^+,p^+\rangle &= \langle 1,+1||\tfrac{1}{2},+\tfrac{1}{2}\rangle|\tfrac{1}{2},+\tfrac{1}{2}\rangle = 1,\\
\mathfrak{M}_b \propto \langle d,\pi^0|p^+,n^0\rangle &= \langle 1,0||\tfrac{1}{2},+\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle,\\
&= \langle 1,0|\left(\frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,0\rangle\right)\right) = \frac{1}{\sqrt{2}},\\
\mathfrak{M}_c \propto \langle d,\pi^-|n^0,n^0\rangle &= \langle 1,-1||\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle = 1.
\end{aligned}$$

Isospin

Applications

- Within the quark-model,

$$|u\rangle = |\tfrac{1}{2}, +\tfrac{1}{2}\rangle, \qquad |d\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

- proceed as before, with:

$$\begin{aligned}
|\pi^+\rangle &= |1,+1\rangle = |u\rangle \otimes |\bar{d}\,\rangle, \\
|\pi^0\rangle &= |1,0\rangle \quad = \frac{1}{\sqrt{2}}|u\rangle \otimes |\bar{u}\rangle + \frac{1}{\sqrt{2}}|d\rangle \otimes |\bar{d}\,\rangle, \\
|\pi^-\rangle &= |1,-1\rangle = |d\rangle \otimes |\bar{u}\rangle, \\
|\eta^0\rangle &\approx |0,0\rangle \quad = \frac{1}{\sqrt{2}}|u\rangle \otimes |\bar{u}\rangle - \frac{1}{\sqrt{2}}|d\rangle \otimes |\bar{d}\,\rangle.
\end{aligned}$$

$$\begin{aligned}
|\Delta^{++}\rangle &= |u\rangle \otimes |u\rangle \otimes |u\rangle = |\tfrac{3}{2}, +\tfrac{3}{2}\rangle, \\
|\Delta^{+}\rangle &= |u\rangle \otimes |u\rangle \otimes |d\rangle = |\tfrac{3}{2}, +\tfrac{1}{2}\rangle, \quad |p^+\rangle = |\tfrac{1}{2}, +\tfrac{1}{2}\rangle, \\
|\Delta^{0}\rangle &= |u\rangle \otimes |d\rangle \otimes |d\rangle = |\tfrac{3}{2}, -\tfrac{1}{2}\rangle, \quad |n^0\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle, \\
|\Delta^{-}\rangle &= |d\rangle \otimes |d\rangle \otimes |d\rangle = |\tfrac{3}{2}, -\tfrac{3}{2}\rangle.
\end{aligned}$$

Isospin

Applications

- Six elastic collisions:

$$(a)\ \pi^+ + p^+ \to \pi^+ + p^+, \quad (b)\ \pi^0 + p^+ \to \pi^0 + p^+, \quad (c)\ \pi^- + p^+ \to \pi^- + p^+,$$
$$(d)\ \pi^+ + n^0 \to \pi^+ + n^0, \quad (e)\ \pi^0 + n^0 \to \pi^0 + n^0, \quad (f)\ \pi^- + n^0 \to \pi^- + n^0,$$

- Four inelastic collisions:

$$(g)\ \pi^+ + n^0 \to \pi^0 + p^+, \qquad (h)\ \pi^0 + p^+ \to \pi^+ + n^0,$$
$$(i)\ \pi^0 + n^0 \to \pi^- + p^+, \qquad (j)\ \pi^- + p^+ \to \pi^0 + n^0.$$

- Since $I(\pi) = 1$ and $I(p^+,n^0) = ½$, their sum determines the two possible "reduced matrix elements" (Wigner-Eckardt Thm.!)

$$\mathfrak{M}_{3/2} \text{ and } \mathfrak{M}_{1/2}$$

Isospin

Applications

- So,

$$
\begin{aligned}
\pi^+ + p^+ : &\quad |1,1\rangle|\tfrac{1}{2},+\tfrac{1}{2}\rangle = |\tfrac{3}{2},+\tfrac{3}{2}\rangle,\\
\pi^0 + p^+ : &\quad |1,0\rangle|\tfrac{1}{2},+\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}|\tfrac{3}{2},+\tfrac{1}{2}\rangle - \tfrac{1}{\sqrt{3}}|\tfrac{1}{2},+\tfrac{1}{2}\rangle,\\
\pi^- + p^+ : &\quad |1,-1\rangle|\tfrac{1}{2},+\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}|\tfrac{3}{2},-\tfrac{1}{2}\rangle - \sqrt{\tfrac{2}{3}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle,\\
\pi^+ + n^0 : &\quad |1,1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{3}}|\tfrac{3}{2},+\tfrac{1}{2}\rangle + \sqrt{\tfrac{2}{3}}|\tfrac{1}{2},+\tfrac{1}{2}\rangle,\\
\pi^0 + n^0 : &\quad |1,0\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \sqrt{\tfrac{2}{3}}|\tfrac{3}{2},-\tfrac{1}{2}\rangle + \tfrac{1}{\sqrt{3}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle,\\
\pi^- + n^0 : &\quad |1,-1\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle = |\tfrac{3}{2},-\tfrac{3}{2}\rangle.
\end{aligned}
$$

- then, e.g.,

$$(a)\ \pi^+ + p^+ \to \pi^+ + p^+ \quad \leftrightarrow \quad \mathfrak{M}_a = \langle\tfrac{3}{2},+\tfrac{3}{2}||\tfrac{3}{2},+\tfrac{3}{2}\rangle \times \mathfrak{M}_{3/2} = \mathfrak{M}_{3/2},$$

$$(c)\ \pi^- + p^+ \to \pi^- + p^+$$

$$
\begin{aligned}
\mapsto \quad \mathfrak{M}_c &= \Big(\tfrac{1}{\sqrt{3}}\langle\tfrac{3}{2},-\tfrac{1}{2};\mathfrak{f}_3| - \sqrt{\tfrac{2}{3}}\langle\tfrac{1}{2},-\tfrac{1}{2};\mathfrak{f}_1|\Big)\Big(\tfrac{1}{\sqrt{3}}|\tfrac{3}{2},-\tfrac{1}{2};\mathfrak{f}_3\rangle - \sqrt{\tfrac{2}{3}}|\tfrac{1}{2},-\tfrac{1}{2};\mathfrak{f}_1\rangle\Big),\\
&= \tfrac{1}{3}\langle\tfrac{3}{2},-\tfrac{1}{2};\mathfrak{f}_3||\tfrac{3}{2},-\tfrac{1}{2};\mathfrak{f}_3\rangle + \tfrac{2}{3}\langle\tfrac{1}{2},-\tfrac{1}{2};\mathfrak{f}_1||\tfrac{1}{2},-\tfrac{1}{2};\mathfrak{f}_1\rangle = \tfrac{1}{3}\mathfrak{M}_{3/2} + \tfrac{2}{3}\mathfrak{M}_{1/2},
\end{aligned}
$$

Generalization of Isospin

Flavor SU(3)

- Pauli matrices: a basis of traceless Hermitian 2×2 matrices
  - span the $\mathfrak{su}(2)$ algebra, generate the SU(2) group
- For the $\mathfrak{su}(3)$ algebra, we need traceless Hermitian 3×3 matrices

$$\boxed{\mathfrak{su}(2)\;\; \boldsymbol{\lambda}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\lambda}_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\lambda}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},} \quad \boldsymbol{\lambda}_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{\lambda}_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\lambda}_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{\lambda}_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \boldsymbol{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

- Gell-Mann's matrices.

$$\left[\, Q_a \,,\, Q_b \,\right] = i\, f_{ab}{}^{c}\, Q_c .$$

$$f_{123} = 1, \quad f_{458} = f_{678} = \frac{\sqrt{3}}{2},$$

$$f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2},$$

$$\mathrm{Tr}(\boldsymbol{\lambda}^a \boldsymbol{\lambda}^b) = 2\delta^{ab}.$$

Generalization of Isospin

Flavor SU(3)

- The $SU_f(3)$ triplet: $t^\alpha = (u, d, s)$, then we have

$$\mathbf{1} \simeq t \qquad \mathbf{3} \simeq t^\alpha \qquad \mathbf{3}^* \simeq t_\alpha = \tfrac{1}{2}\varepsilon_{\alpha\beta\gamma} t^{[\beta\gamma]} \qquad \alpha, \beta, \gamma, \ldots = 1, 2, 3,$$

$$\mathbf{6} \simeq t^{(\alpha\beta)}, \qquad \mathbf{6}^* \simeq t_{(\alpha\beta)} \qquad \mathbf{8} \simeq t^\alpha{}_\beta,\ t^\alpha{}_\alpha \equiv 0 \qquad \mathbf{10} \simeq t^{(\alpha\beta\gamma)},$$

- … and so on.
- 2-quark states are then given by **3⊗3 = 6**$_{\mathrm{S}}$**⊕3***$_{\mathrm{A}}$:

$$t^\alpha\, s^\beta = t^{(\alpha} s^{\beta)} + t^{[\alpha} s^{\beta]}, \qquad \begin{cases} t^{(\alpha} s^{\beta)} & := & \frac{1}{2}\left(t^\alpha s^\beta + t^\beta s^\alpha\right), \\ t^{[\alpha} s^{\beta]} & := & \frac{1}{2}\left(t^\alpha s^\beta - t^\beta s^\alpha\right); \end{cases}$$

- and 3-quark states by **6⊗3 = 10⊕8**:

$$t^{(\alpha\beta)}\, s^\gamma = t^{(\alpha\beta} s^{\gamma)} + \tfrac{4}{3}\, t^{(\alpha[b)} s^{\gamma]},$$

$$t^{(\alpha\beta} s^{\gamma)} := \tfrac{1}{3}\left(t^{(\alpha\beta)} s^\gamma + t^{(\beta\gamma)} s^\alpha + t^{(\gamma\alpha)} s^\beta\right),$$

$$t^{(\alpha[\beta)} s^{\gamma]} := \tfrac{1}{4}\left(\left(t^{(\alpha\beta)} s^\gamma - t^{(\alpha\gamma)} s^\beta\right) + \left(t^{(\beta\alpha)} s^\gamma - t^{(\beta\gamma)} s^\alpha\right)\right),$$

$$= \tfrac{1}{4}\left(2t^{(\alpha\beta)} s^\gamma - t^{(\alpha\gamma)} s^\beta - t^{(\beta\gamma)} s^\alpha\right)$$

Generalization of Isospin

Flavor SU(3)

- Note, the "funny" **8** is annihilated by antisymmetrization:

$$t^{(\alpha[\beta)}s^{\gamma]}\,\varepsilon_{\alpha\beta\gamma} \equiv 0;$$

- Also, antiquark-quark states form $\mathbf{3}^*\otimes\mathbf{3} = \mathbf{8}\oplus\mathbf{1}$:

$$t_\alpha\, s^\beta = \left(t_\alpha s^\beta - \tfrac{1}{3}\delta^\beta_\alpha (t_\gamma s^\gamma)\right) + \tfrac{1}{3}\delta^\beta_\alpha (t_\gamma s^\gamma).$$

- The two **8**'s are related:

$$\tfrac{4}{3}\, t^{(\alpha[\beta)}s^{\gamma]}\,\varepsilon_{\beta\gamma\delta} = t^{(\alpha\beta)}s^\gamma\,\varepsilon_{\beta\gamma\delta} =: (t^{(\cdot\cdot)}s^\cdot)^\alpha{}_\delta \qquad : \qquad \delta_\alpha{}^\delta\,(t^{(\cdot\cdot)}s^\cdot)^\alpha{}_\delta \equiv 0$$

$$(t^{(\cdot\cdot)}s^\cdot)^\alpha{}_\delta\,\varepsilon^{\beta\gamma\delta} = t^{(\alpha\beta)}s^\gamma - t^{(\alpha\gamma)}s^\beta, \qquad \tfrac{2}{3}(t^{(\cdot\cdot)}s^\cdot)^{(\alpha}{}_\delta\,\varepsilon^{\beta)\gamma\delta} = \tfrac{4}{3}\,t^{(\alpha[\beta)}s^{\gamma]}$$

- Finally, $\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} = (\mathbf{6}_S\oplus\mathbf{3}_A)\otimes\mathbf{3} = (\mathbf{10}_S\oplus\underbrace{\mathbf{8})\oplus(\mathbf{8}}_{\text{mixed symmetry}}\oplus\mathbf{1}_A)$.

Generalization of Isospin

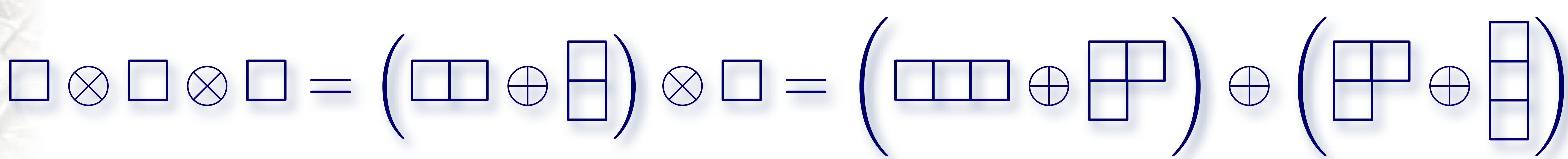
Flavor SU(3)

- There are really four linearly independent choices:

			$\alpha\beta\gamma$	$\alpha\gamma\beta$	$\gamma\alpha\beta$	$\gamma\beta\alpha$	$\beta\gamma\alpha$	$\beta\alpha\gamma$
10	:	$t^{(\alpha\beta\gamma)}$	$+1$	$+1$	$+1$	$+1$	$+1$	$+1$
8	:	$t^{(\alpha[\beta)\gamma]}$	$+2$	-1	-1	-1	-1	$+2$
8	:	$t^{[\alpha(\beta]\gamma)}$	$+2$	$+1$	-1	$+1$	-1	-2
1	:	$t^{[\alpha\beta\gamma]}$	$+1$	-1	$+1$	-1	$+1$	-1

- proving that

$$\underbrace{t^{\alpha\beta\gamma}}_{\text{no symmetry}} = \underbrace{t^{(\alpha\beta\gamma)}}_{\mathbf{10}_S} + \underbrace{t^{(\alpha[\beta)\gamma]}}_{\mathbf{8}} + \underbrace{t^{[\alpha(\beta]\gamma)}}_{\mathbf{8}} + \underbrace{t^{[\alpha\beta\gamma]}}_{\mathbf{1}_A}$$



$\mathbf{10}_S$ $\quad \mathbf{8} \quad \mathbf{8} \quad \mathbf{1}_A$

Quark Bound States

Light Meson Masses

- Charmonium (and later bottomonium) are well modeled by
- …and non-relativistic "equal mass modified" H-atom model.
- Turn to light mesons.
- Only (u, d) at first & isospin $SU(2)$.

$$\{t^1, t^2\} = \{u,\ d\}, \quad \Rightarrow \quad \{t_1, t_2\} = \{\overline{u},\ \overline{d}\}.$$

$$t_1 = \varepsilon_{12} t^2 = t^2 \quad \Rightarrow \quad |\overline{u}\rangle = |d\rangle,$$

$$t_2 = \varepsilon_{21} t^1 = -t^1 \quad \Rightarrow \quad |\overline{d}\rangle = -|u\rangle.$$

$$\begin{aligned}\big\{|\overline{u}\rangle,\ |\overline{d}\rangle\big\} \otimes \big\{|u\rangle,\ |d\rangle\big\} &= \big\{|\tfrac{1}{2}, -\tfrac{1}{2}\rangle, -|\tfrac{1}{2}, \tfrac{1}{2}\rangle\big\} \otimes \big\{|\tfrac{1}{2}, +\tfrac{1}{2}\rangle, |\tfrac{1}{2}, -\tfrac{1}{2}\rangle\big\}, \\ &= \big\{|1, \pm 1\rangle,\ |1, 0\rangle\big\} \oplus \big\{|0, 0\rangle\big\} = \big\{|\pi^{\pm}\rangle,\ |\pi^0\rangle\big\} \oplus \big\{|\eta\rangle\big\},\end{aligned}$$

Quark Bound States

Light Meson Masses

- In particular,

$$
\begin{aligned}
|1,+1\rangle &= |\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle && = -|\bar{d}\rangle|u\rangle, \\
|1,0\rangle &= \frac{1}{\sqrt{2}}\Big(|\tfrac{1}{2},+\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle + |\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},+\tfrac{1}{2}\rangle\Big) && = \frac{1}{\sqrt{2}}\Big(-|\bar{d}\rangle|d\rangle + |\bar{u}\rangle|u\rangle\Big), \\
|1,-1\rangle &= |\tfrac{1}{2},+\tfrac{1}{2}\rangle|\tfrac{1}{2},+\tfrac{1}{2}\rangle && = |\bar{u}\rangle|d\rangle, \\
|0,0\rangle &= \frac{1}{\sqrt{2}}\Big(|\tfrac{1}{2},+\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle - |\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},+\tfrac{1}{2}\rangle\Big) && = \frac{1}{\sqrt{2}}\Big(-|\bar{d}\rangle|d\rangle - |\bar{u}\rangle|u\rangle\Big),
\end{aligned}
$$

$$
\left\{
\begin{aligned}
|\pi^+\rangle &= -|\bar{d}\,u\rangle, \\
|\pi^0\rangle &= \frac{1}{\sqrt{2}}\big(|\bar{u}\,u\rangle - |\bar{d}\,d\rangle\big), \\
|\pi^-\rangle &= |\bar{u}\,d\rangle,
\end{aligned}
\right.
\quad \text{and} \quad |\eta\rangle = -\frac{1}{\sqrt{2}}\big(|\bar{u}\,u\rangle + |\bar{d}\,d\rangle\big).
$$

- There is no symmetry between a quark and an antiquark.
- In fact, however, the η-particle also contains a strange-antistrange component.

Quark Bound States

Light Meson Masses

- Mesons with u, d and s:

$$\pi^+ = (\bar{d}u), \quad \pi^- = (\bar{u}d), \quad \pi^0 = \frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d), \quad \eta = \frac{1}{\sqrt{6}}(\bar{u}u + \bar{d}d - 2\bar{s}s),$$

$$K^+ = (\bar{s}u), \quad K^0 = (\bar{s}d), \quad \overline{K}^0 - = (\bar{d}s), \quad K^- = (\bar{u}s), \quad \eta' = \frac{1}{\sqrt{3}}(\bar{u}u + \bar{d}d + \bar{s}s).$$

- Recall:

$$\boldsymbol{\lambda}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\lambda}_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\lambda}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\lambda}_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{\lambda}_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \quad \boldsymbol{\lambda}_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \boldsymbol{\lambda}_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad \boldsymbol{\lambda}_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

- In turn, however, amongst the "P-state" $(\ell = 1)$ mesons

$$\omega \neq \frac{1}{\sqrt{6}}(\bar{u}u + \bar{d}d - 2\bar{s}), \qquad \text{but} \qquad \omega \approx \frac{1}{\sqrt{2}}(\bar{u}u + \bar{d}d),$$

$$\phi \neq \frac{1}{\sqrt{3}}(\bar{u}u + \bar{d}d + \bar{s}), \qquad \text{but} \qquad \phi \approx (\bar{s}s).$$

Quark Bound States

Light Meson Masses

- Since pseudo-scalar (S-state) and vector (P-state) mesons differ only in the relative orientation of spins,
- …the spin-spin coupling should provide the dominant "correction" to the meson masses:

$$M(\text{meson}) \approx m_q + m_{\overline{q}} + \frac{A}{m_q m_{\overline{q}}} \left\langle \mathbf{S}_q \cdot \mathbf{S}_{\overline{q}} \right\rangle,$$

- …where A is fitted from experimental data, and

$$\vec{S}_q \cdot \vec{S}_{\overline{q}} = \begin{cases} \frac{1}{4}\hbar^2, & \text{for } S = 1 \text{ (vector mesons)}, \\ -\frac{3}{4}\hbar^2, & \text{for } S = 0 \text{ (pseudo-scalar mesons)}. \end{cases}$$

Meson	Comp.	Exp.
π	140	138
K	485	496
η	559	549
η'	303	958

Meson	Comp.	Exp.
ρ	780	776
ω	780	783
K^*	896	892
ϕ	1 032	1 020

Quark Bound States

Light Baryon Masses

- Being 3-quark states, baryon classification is harder.
- For S-states (both orbital angular momenta = 0), the baryon spin stems from quark spins, added.
- Use the basis

$$\left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle_{[12]} = \frac{1}{\sqrt{2}}\big(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\big), \qquad \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle_{[12]} = \frac{1}{\sqrt{2}}\big(|\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle\big);$$

$$\left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle_{[23]} = \frac{1}{\sqrt{2}}\big(|\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\uparrow\rangle\big), \qquad \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle_{[23]} = \frac{1}{\sqrt{2}}\big(|\downarrow\uparrow\downarrow\rangle - |\downarrow\downarrow\uparrow\rangle\big),$$

- and note that

$$\left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle_{[13]} = \frac{1}{\sqrt{2}}\big(|\uparrow\uparrow\downarrow\rangle - |\downarrow\uparrow\uparrow\rangle\big) = \left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle_{[12]} + \left|\tfrac{1}{2},+\tfrac{1}{2}\right\rangle_{[23]},$$

- and then factorize

$$\Psi(\text{baryon}) = \Psi(\vec{r},t)\,\chi(\text{spin})\,\chi(\text{“flavor”})\,\chi(\text{color}).$$

Quark Bound States

Light Baryon Masses

- The totally symmetric spin states are easy:

$$|\tfrac{3}{2},+\tfrac{3}{2}\rangle = |\uparrow\uparrow\uparrow\rangle, \qquad |\tfrac{3}{2},+\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}\left(|\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\rangle\right),$$

$$|\tfrac{3}{2},-\tfrac{3}{2}\rangle = |\downarrow\downarrow\downarrow\rangle, \qquad |\tfrac{3}{2},-\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}\left(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle\right),$$

- The mixed states are tricky:

$$\chi_{[12]}(\text{spin})\,\chi_{[12]}(\text{“flavor”}) + \chi_{[13]}(\text{spin})\,\chi_{[13]}(\text{“flavor”}) + \chi_{[23]}(\text{spin})\,\chi_{[23]}(\text{“flavor”})$$

- So then, either

$$\Psi(\text{baryon}) = \underset{\text{S}}{\Psi(\vec{r},t)}\;\underset{\text{S}}{\chi(\text{spin})}\;\underset{\text{S} = \mathbf{10}}{\chi(\text{“flavor”})}\;\underset{\text{A}}{\chi(\text{color})}.$$

- or

$$\Psi(\text{baryon}) = \underset{\text{S}}{\Psi(\vec{r},t)}\;\underbrace{\underset{\text{M}}{\chi(\text{spin})}\;\underset{\text{M} = \mathbf{8}}{\chi(\text{“flavor”})}}_{\text{symmetric}}\;\underset{\text{A}}{\chi(\text{color})}.$$

Quark Bound States

Light Baryon Masses

- Again, the distinction between the bayon **10**-plet and **8**-plet stem from differences in spin-orientations, so:

$$M(\text{baryon}) \approx m_1 + m_2 + m_3 + A' \sum_{i \neq j} \frac{1}{m_i m_j} \langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle ,$$

- Since $m_u \approx m_d < m_s$,

$$M(\Delta) \approx 3m_u + \frac{3A'\hbar^2}{4m_u^2}, \qquad \text{and} \qquad M(\Omega^-) \approx 3m_s + \frac{3A'\hbar^2}{4m_s^2},$$

$$M(\Sigma^*) \approx 2m_u + m_s + \frac{A'\hbar^2}{4}\left(\frac{1}{m_u^2} + \frac{2}{m_u m_s}\right),$$

$$M(\Xi^*) \approx m_u + 2m_s + \frac{A'\hbar^2}{4}\left(\frac{2}{m_u m_s} + \frac{1}{m_s^2}\right).$$

- works well for the **10**-plet.

Quark Bound States

Light Baryon Masses

- Use also

$$\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_1 \cdot \vec{S}_3 + \vec{S}_2 \cdot \vec{S}_3 = \tfrac{1}{2}\Big((\vec{S}_1 + \vec{S}_2 + \vec{S}_3)^2 - S_1^2 - S_2^2 - S_3^2\Big),$$

$$\langle \vec{S}_1 \cdot \vec{S}_2 + \vec{S}_1 \cdot \vec{S}_3 + \vec{S}_2 \cdot \vec{S}_3 \rangle = \begin{cases} \tfrac{3}{4}\hbar^2 & \text{for spin-}3/2\text{ 10-plet,} \\ -\tfrac{3}{4}\hbar^2 & \text{for spin-}1/2\text{ 8-plet.} \end{cases}$$

- and obtain:

$$M(p^+, n^0) \approx 3m_u - \frac{3A'\hbar^2}{4m_u^2},$$

$$M(\Lambda) \approx 2m_u + m_s - \frac{3A'\hbar^2}{4m_u^2},$$

$$M(\Sigma) \approx 2m_u + m_s + \frac{A'\hbar^2}{4}\left(\frac{1}{m_u^2} - \frac{4}{m_u m_s}\right),$$

$$M(\Xi) \approx 2m_u + m_s + \frac{A'\hbar^2}{4}\left(\frac{1}{m_s^2} - \frac{4}{m_u m_s}\right).$$

Quark Bound States

Light Baryon Masses

- Thus, finally:

Baryon	Comp.	Exp.
p^+, n^0	939	939
Λ	1 116	1 114
Σ	1 179	1 193
Ξ	1 327	1 318

Baryon	Comp.	Exp.
Δ	1 239	1 232
Σ^*	1 381	1 384
Ξ^*	1 529	1 533
Ω	1 682	1 672

- Pretty good …
- … and not just for complicated 3-body bound states!!

Quark Bound States

Light Baryon Magnetic Moments

- Straightforwardly, to first order,

$$\vec{\mu}(\text{baryon}) = \vec{\mu}_{(1)} + \vec{\mu}_{(2)} + \vec{\mu}_{(3)}.$$

$$\langle \mu_3 \rangle = \left\langle \frac{q}{mc} S_3 \right\rangle = \pm \frac{q\hbar}{2mc}, \qquad \begin{cases} \mu_u := \langle \mu_{(u),3} \rangle = \pm \dfrac{e\hbar}{3m_u c}, \\ \mu_d := \langle \mu_{(d),3} \rangle = \mp \dfrac{e\hbar}{6m_d c}, \\ \mu_s := \langle \mu_{(s),3} \rangle = \mp \dfrac{e\hbar}{6m_s c}, \end{cases}$$

- and then

$$\langle \mu_3(\text{baryon}) \rangle = \frac{2}{\hbar} \sum_{i=1}^{3} \langle \text{baryon} | \mu_i \, S_{(i),3} | \text{baryon} \rangle .$$

- Must use the wave-functions for the baryons …
- … the simpler ones for the **10**-plet,
- … the more complicated for the **8**-plet.

Quark Bound States

Light Baryon Magnetic Moments

- This produces:

Baryon	$\langle \mu_3 \rangle$	Comp.	Exp.
p^+	$\frac{1}{3}(4\mu_u - \mu_d)$	2.79	2.793
n^0	$\frac{1}{3}(4\mu_d - \mu_u)$	-1.86	-1.913
Ξ^0	$\frac{1}{3}(4\mu_s - \mu_u)$	-1.40	-1.253
Ξ^-	$\frac{1}{3}(4\mu_u - \mu_s)$	-0.47	-0.69
Λ^0	μ_s	-0.58	-0.61
Σ^+	$\frac{1}{3}(4\mu_u - \mu_s)$	2.68	2.33
Σ^0	$\frac{1}{3}(2\mu_u + \mu_d - \mu_s)$	0.82	
Σ^-	$\frac{1}{3}(4\mu_d - \mu_s)$	-1.05	-1.41

Thanks!

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