(Fundamental) Physics of Elementary Particles

Perturbation Theory in Picutres: Feynman Diagrams

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Fundamental Physics of Elementary Particles

Program

- Perturbation Theory in Pictures
  - Stationary perturbation theory in Quantum Mechanics
  - Feynman diagrams in perturbative Quantum Field Theory
    - The Heisenberg Zone — and What's Going On
    - A toy-model work-out
- A Few Stray Comments
  - An overview of the subject topic
  - A review of all conservation laws

Perturbation Picturebook

Quantum Mechanics: Warm-Up Calisthenics

- Consider a quantum-mechanical system where the Hamiltonian differs from a well-known one by H'.
- We want to find solutions to:

$$H\,|n\rangle = E_n\,|n\rangle, \qquad H \;:=\; H_0 + \lambda\, H'$$

- … where λ is a bookkeeping parameter.
- Given that we *do* know:

$$H_0|n;0\rangle = E_n^{(0)}|n;0\rangle, \qquad \begin{cases} \langle n;0|n';0\rangle &= \delta_{n,n'}, \\ \sum_n |n;0\rangle\langle n;0| &= \mathbb{1}, \end{cases}$$

- … we expand as a power-series in λ.

$$E_n = \sum_{k=0}^{\infty} \lambda^k\, E_n^{(k)}, \qquad |n\rangle = \sum_{k=0}^{\infty} \lambda^k\,|n;k\rangle$$

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Quantum Mechanics: Warm-Up Calisthenics

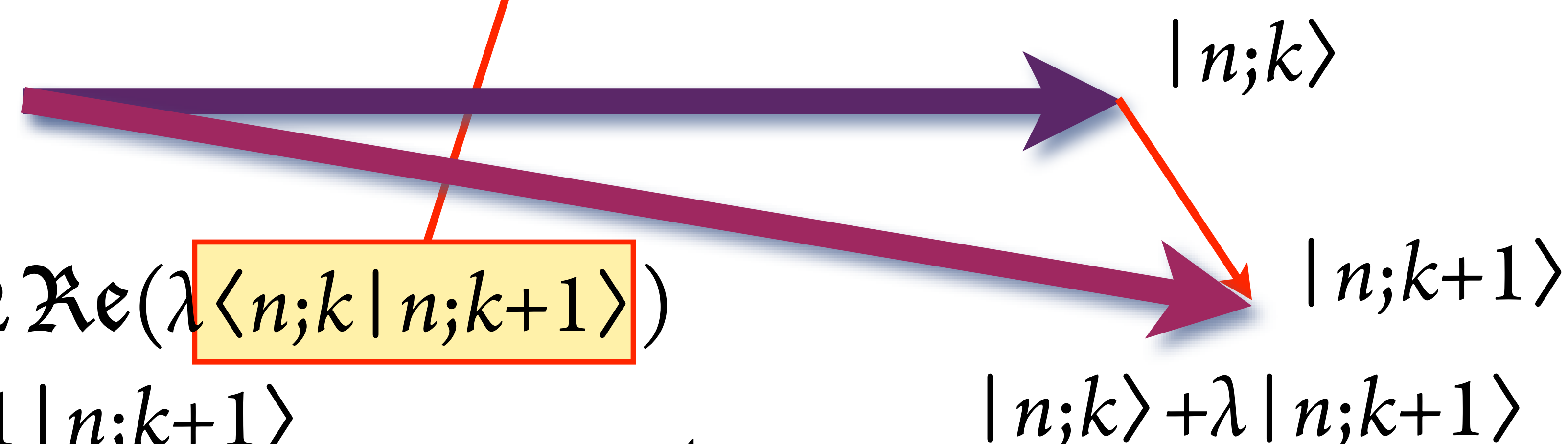
- We demand that order-by-order, the bases are orthonormalized (*always* doable, by Gram–Schmidt!)

$$\langle m;k|n;k\rangle = \delta_{mn}, \quad \forall m,n,$$

- but also that the "same" kets from different orders of perturbation are orthogonal:

$$\langle n;k|n;\ell\rangle = \delta_{k\ell}, \quad \forall k,\ell.$$

- Why? For consistency of the normalization!



$$\langle n;k|n;k\rangle + 2\,\mathfrak{Re}\big(\lambda\langle n;k|n;k{+}1\rangle\big) + |\lambda|^2\langle n;k{+}1|n;k{+}1\rangle$$

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Quantum Mechanics: Warm-Up Calisthenics

- Now introduce specially weighted projectors:

$$\widehat{\Pi}_n^{\alpha} \;:=\; \sum_{m\neq n} \frac{|m;0\rangle\langle m;0|}{\big(E_n^{(0)}-E_m^{(0)}\big)^{\alpha}}, \qquad \text{so} \qquad \widehat{\Pi}_n^{\alpha}\,\widehat{\Pi}_n^{\beta} = \widehat{\Pi}_n^{\alpha+\beta}$$

- They project back to the well-known, initial basis, ***away from*** $|n;0\rangle$, and curiously normalized.
- Note the no-degeneracy assumption!
- With this notation (≈Cohen-Tannoudji, Diu, Laloë),

$$\Big[H_0+\lambda H'\Big]\Big(\sum_{k=0}^{\infty}\lambda^k\,|n;k\rangle\Big) = \Big(\sum_{k'=0}^{\infty}\lambda^{k'}\,E_n^{(k')}\Big)\Big(\sum_{k''=0}^{\infty}\lambda^{k''}\,|n;k''\rangle\Big)$$

- … from where one obtains "order-by-order equations":

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Quantum Mechanics: Warm-Up Calisthenics

- The "order-by-order" equations:

$$|n;k\rangle = \widehat{\Pi}_n^1\, H'|n;k{-}1\rangle - \sum_{i=1}^{k-1} E_n^{(i)}\, \widehat{\Pi}_n^1\, |n;k-i\rangle, \qquad k>0,$$
$$E_n^{(k)} = \langle n;0\big|H'\big|n;k{-}1\rangle.$$

- The first few of these are easy:

$$E_n^{(1)} = \langle n;0|H'|n;0\rangle,$$
$$|n;1\rangle = \widehat{\Pi}_n^1\, H'|n;0\rangle,$$
$$E_n^{(2)} = \langle n;0\big|H'\,\widehat{\Pi}_n^1\, H'\big|n;0\rangle$$
$$|n;2\rangle = \widehat{\Pi}_n^1 (H' - E_n^{(1)})|n;1\rangle,$$
$$= \widehat{\Pi}_n^1\, H'\,\widehat{\Pi}_n^1\, H'|n;0\rangle - \widehat{\Pi}_n^1\,\widehat{\Pi}_n^1\, H'|n;0\rangle\langle n;0\big|H'\big|n;0\rangle,$$
$$= \big[\,\widehat{\Pi}_n^1\, H'\,\widehat{\Pi}_n^1 - \widehat{\Pi}_n^2\, H'|n;0\rangle\langle n;0|\,\big]\, H'|n;0\rangle,$$

…a subtraction!

Perturbation Picturebook

Quantum Mechanics: Warm-Up Calisthenics

- Soon enough, the complications grow:

$$\begin{aligned} E_n^{(3)} &= \langle n;0 | H' | n;2 \rangle \\ &= \langle n;0|H' \left[\widehat{\Pi}_n^1 \, H' \, \widehat{\Pi}_n^1 - \widehat{\Pi}_n^2 \, H' |n;0\rangle\langle n;0| \right] H' |n;0\rangle \\ &= \langle n;0|H' \, \widehat{\Pi}_n^1 \, H' \, \widehat{\Pi}_n^1 \, H'|n;0\rangle - \langle n;0|H' \, \widehat{\Pi}_n^2 \, H'|n;0\rangle\langle n;0|H'|n;0\rangle, \end{aligned}$$

- … so the energy correction also requires subtractions …

$$\begin{aligned} |n;3\rangle &= \widehat{\Pi}_n^1 \left((H' - E_n^{(1)})|n;2\rangle - E_n^{(2)}|n;1\rangle \right), \\ &= \widehat{\Pi}_n^1 \, H'|n;2\rangle - \widehat{\Pi}_n^1 \, |n;2\rangle\langle n;0|H'|n;0\rangle - \widehat{\Pi}_n^1 \, |n;1\rangle\langle n;0|H' \, \widehat{\Pi}_n^1 \, H'|n;0\rangle, \\ &= \widehat{\Pi}_n^1 \, H' \, \widehat{\Pi}_n^1 \, H' \, \widehat{\Pi}_n^1 \, H'|n;0\rangle - \widehat{\Pi}_n^1 \, H' \, \widehat{\Pi}_n^2 \, H'|n;0\rangle\langle n;0|H'|n;0\rangle \\ &\quad - \widehat{\Pi}_n^2 \, H' \, \widehat{\Pi}_n^1 \, H'|n;0\rangle\langle n;0|H'|n;0\rangle - \widehat{\Pi}_n^2 \, H'|n;0\rangle\langle n;0|H' \, \widehat{\Pi}_n^1 \, H'|n;0\rangle \\ &\quad - \widehat{\Pi}_n^3 \, H'|n;0\rangle\langle n;0|H'|n;0\rangle^2, \end{aligned}$$

- … and the subtractions multiply combinatorially.
- The *combinatorial* growth gives a clue: *diagrams!*

Perturbation Picturebook

Quantum Mechanics: Warm-Up Calisthenics

- Before we introduce any diagrams, however, consider:

$$
\begin{aligned}
|n;3\rangle = &\ \widehat{\Pi}_n^1\, H'\, \widehat{\Pi}_n^1\, H'\, \widehat{\Pi}_n^1\, H'|n;0\rangle \\
&- \widehat{\Pi}_n^2\, H'\, \widehat{\Pi}_n^1\, H'|n;0\rangle\langle n;0|H'|n;0\rangle \\
&- \widehat{\Pi}_n^1\, H'\, \widehat{\Pi}_n^2\, H'|n;0\rangle\langle n;0|H'|n;0\rangle \\
&- \widehat{\Pi}_n^2\, H'|n;0\rangle\langle n;0|H'\, \widehat{\Pi}_n^1\, H'|n;0\rangle \\
&- \widehat{\Pi}_n^3\, H'|n;0\rangle\langle n;0|H'|n;0\rangle^2,
\end{aligned}
$$

$$
\begin{aligned}
|n;3\rangle = &\ \widehat{\Pi}_n^1\, H'\, \widehat{\Pi}_n^1\, H'\, \widehat{\Pi}_n^1\, H'|n;0\rangle \\
&- \widehat{\Pi}_n^1\, [H']\, \widehat{\Pi}_n^1\, H'\, \widehat{\Pi}_n^1\, H'|n;0\rangle \\
&- \widehat{\Pi}_n^1\, H'\, \widehat{\Pi}_n^1\, [H']\, \widehat{\Pi}_n^1\, H'|n;0\rangle \\
&- \widehat{\Pi}_n^1\, [H'\, \widehat{\Pi}_n^1\, H']\, \widehat{\Pi}_n^1\, H'|n;0\rangle \\
&- \widehat{\Pi}_n^1\, [H']\, \widehat{\Pi}_n^1\, [H']\, \widehat{\Pi}_n^1\, H'|n;0\rangle
\end{aligned}
$$

$$0 \qquad \left\{ \begin{aligned} &- \widehat{\Pi}_n^1\, H'\, \widehat{\Pi}_n^1\, H'\, \widehat{\Pi}_n^1\, [H']|n;0\rangle \\ &- \widehat{\Pi}_n^1\, H'\, \widehat{\Pi}_n^1\, [H'\, \widehat{\Pi}_n^1\, H']|n;0\rangle \end{aligned} \right.$$

?

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Quantum Mechanics: Warm-Up Calisthenics

- Before we introduce any diagrams, however, consider:

$$|n;3\rangle = \widehat{\Pi}^1_n\, H'\, \widehat{\Pi}^1_n\, H'\, \widehat{\Pi}^1_n\, H'|n;0\rangle$$

$$- \widehat{\Pi}^1_n\, [H']\, \widehat{\Pi}^1_n\, H'\, \widehat{\Pi}^1_n\, H'|n;0\rangle$$

$$- \widehat{\Pi}^1_n\, H'\, \widehat{\Pi}^1_n\, [H']\, \widehat{\Pi}^1_n\, H'|n;0\rangle$$

~~$$- \widehat{\Pi}^1_n\, H'\, \widehat{\Pi}^1_n\, H'\, \widehat{\Pi}^1_n\, [H']|n;0\rangle$$~~

$$- \widehat{\Pi}^1_n\, [H']\, \widehat{\Pi}^1_n\, [H']\, \widehat{\Pi}^1_n\, H'|n;0\rangle$$

$$- \widehat{\Pi}^1_n\, [H'\, \widehat{\Pi}^1_n\, H']\, \widehat{\Pi}^1_n\, H'|n;0\rangle$$

~~$$- \widehat{\Pi}^1_n\, H'\, \widehat{\Pi}^1_n\, [H'\, \widehat{\Pi}^1_n\, H']|n;0\rangle$$~~

An observation, not a proof!

The subtractions may in fact be generated from the original expression!

$$\widehat{\Pi}^1_n\, [H']\, \widehat{\Pi}^1_n\, H'\, \widehat{\Pi}^1_n\, H'|n;0\rangle = \widehat{\Pi}^1_n\, \widehat{\Pi}^1_n\, H'\, \widehat{\Pi}^1_n\, H'|n;0\rangle\, \langle n;0|H'|n;0\rangle,$$

$$\widehat{\Pi}^1_n\, [H'\, \widehat{\Pi}^1_n\, H']\, \widehat{\Pi}^1_n\, H'|n;0\rangle = \widehat{\Pi}^1_n\, \widehat{\Pi}^1_n\, H'|n;0\rangle\, \langle n;0|H'\, \widehat{\Pi}^1_n\, H'|n;0\rangle,$$

- … and so on. Now, you try for the 3rd one …

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Quantum Mechanics: Warm-Up Calisthenics

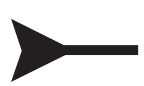
- Recall:

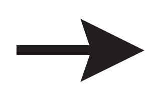
$$\widehat{\Pi}_n^\alpha\,|n;0\rangle = \sum_{m\neq n}\frac{|m;0\rangle\langle m;0|}{(E_n^{(0)}-E_m^{(0)})^\alpha}|n;0\rangle = \sum_{m\neq n}\frac{1}{(E_n^{(0)}-E_m^{(0)})^\alpha}|m;0\rangle\,\underbrace{\langle m;0|n;0\rangle}_{=0\ (\because\ m\neq n)}.$$

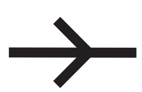
$$|n;k\rangle = (\widehat{\Pi}_n^1\,H')^k|n;0\rangle - \text{all “excisions”}, \qquad k\geqslant 0,$$

$$E_n^{(k)} = \langle n;0|H'(\widehat{\Pi}_n^1\,H')^{k-1}|n;0\rangle - \text{all “excisions”}. \qquad k\geqslant 1,$$

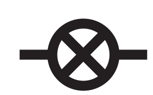
- Now that we have the computations, depict them:

 $= |n;0\rangle,$ *(in)*

 $= \langle n;0|,$ *(out)*

 $= \widehat{\Pi}_n^1,$ *(propagator)*

 $= \widehat{\Pi}_n^2,$ *(2nd order propagator)*

 $= H'.$ *(interaction)*

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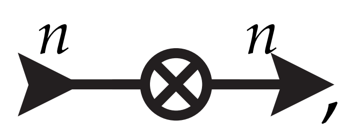
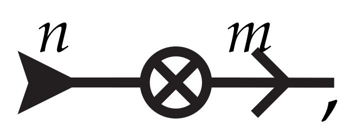
Quantum Mechanics: Warm-Up Calisthenics

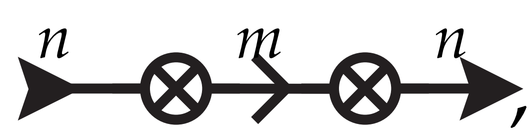
- Soo…

$$\langle n;0|H'|n;0\rangle \overset{(1.89a)}{\longmapsto} E_n^{(1)} =$$ 

,

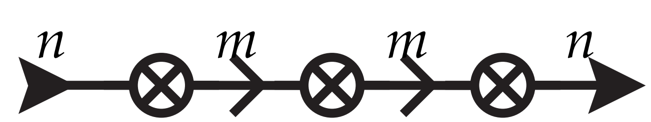
$$\widehat{\Pi}_n^1\, H'|n;0\rangle \overset{(1.89b)}{\longmapsto} |n;1\rangle =$$ 

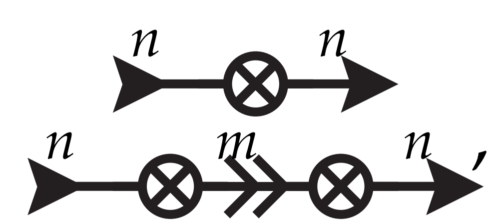
,

$$\langle n;0|H'\,\widehat{\Pi}_n^1\, H'|n;0\rangle \overset{(1.89c)}{\longmapsto} E_n^{(2)} =$$ 

,

$$\langle n;0|H'\,\widehat{\Pi}_n^1\, H'\,\widehat{\Pi}_n^1\, H'|n;0\rangle - \langle n;0|H'\,\widehat{\Pi}_n^2\, H'|n;0\rangle\langle n;0|H'|n;0\rangle$$

$$\overset{(1.89e)}{\longmapsto} E_n^{(3)} =$$ 

$-$ 

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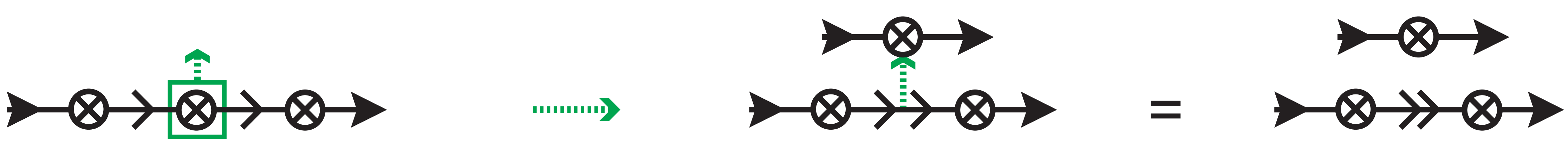
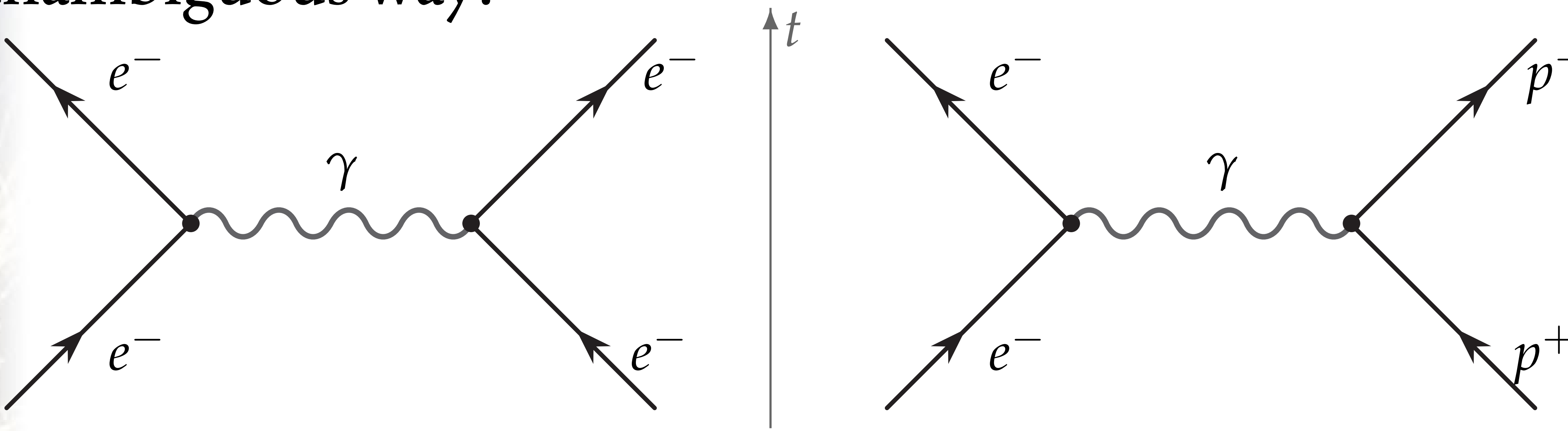


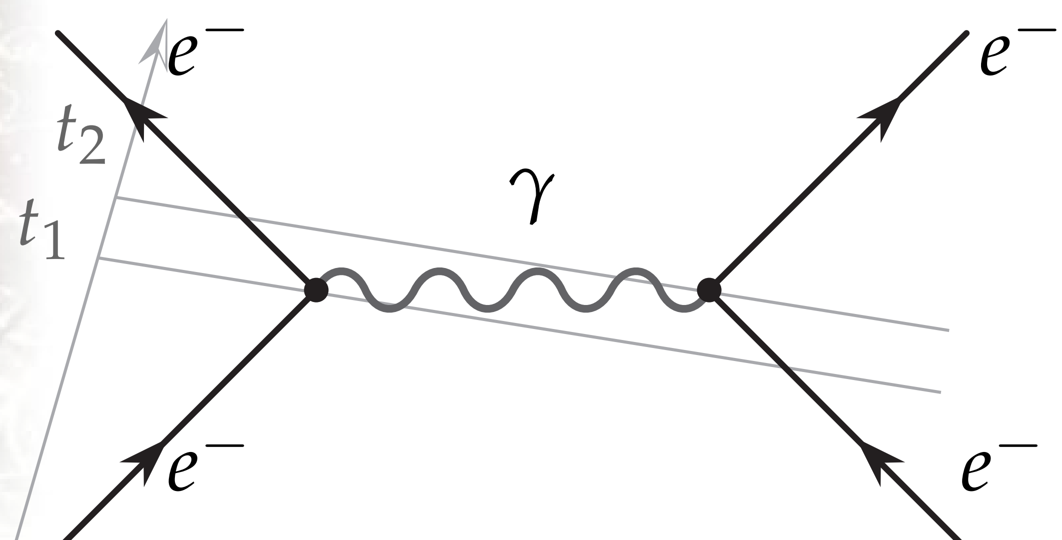
Perturbation Picturebook

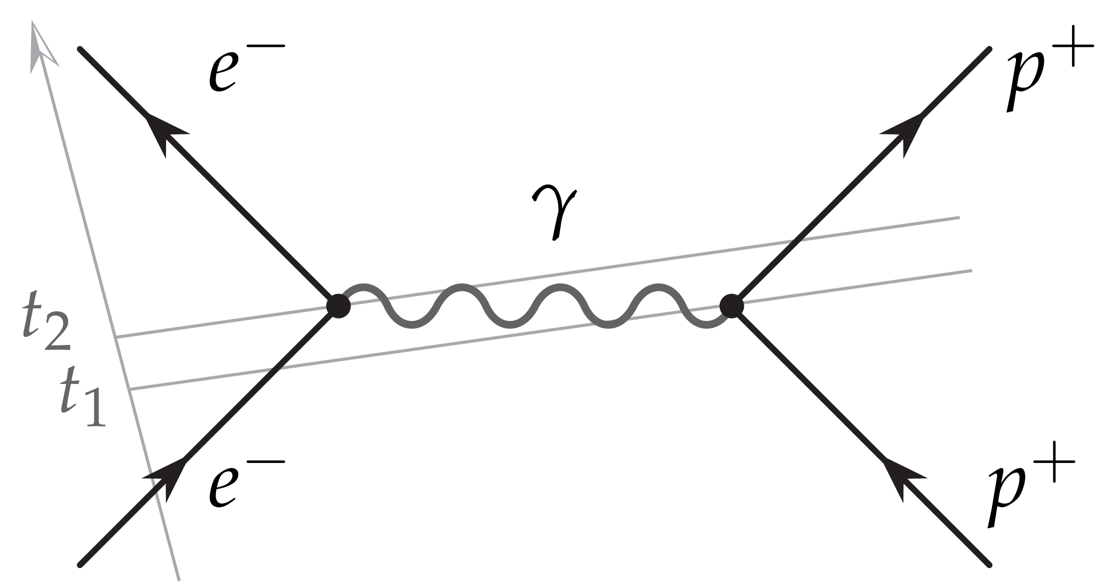
Spacetime Processes (QM ☛ QFT)

- The general idea is to depict physical processes, in a 1–1 unambiguous way:



- … but not literally!

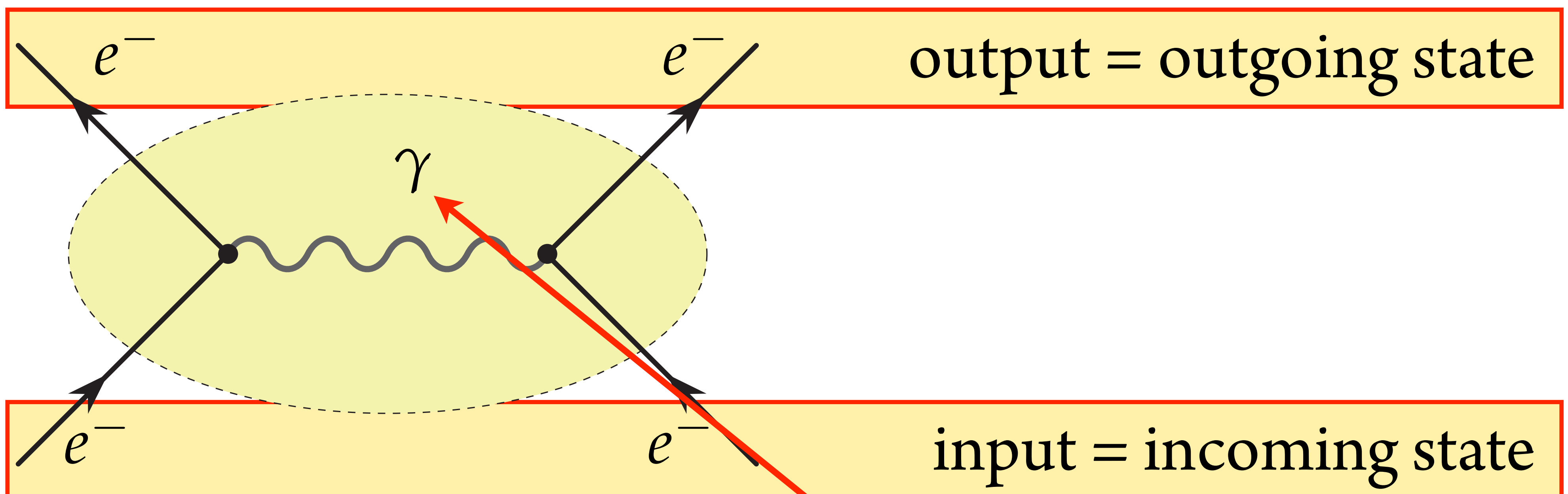




Perturbation Picturebook

Spacetime Processes (QM ☛ QFT)

- The "process" is whatever takes "input" ⟶ "output":



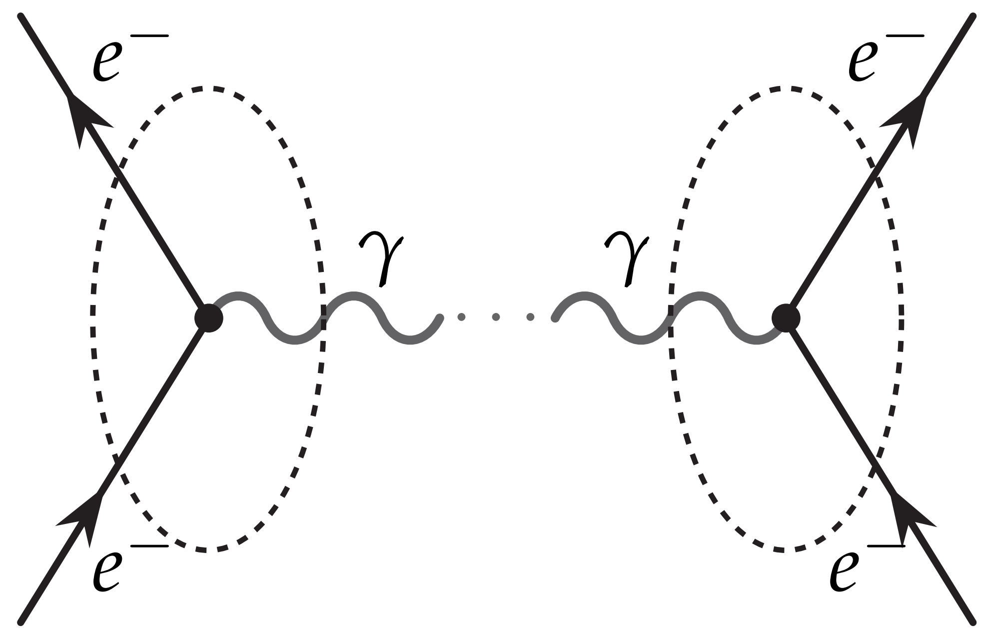
- In this case, it is a 1-photon electromagnetic interaction (scattering) of an electron with another electron.
- What ***is*** observed are the incoming two particles in the incoming 2-particle state, and the outgoing two particles in the outgoing 2-particle state.

Not the exchanged γ!

Perturbation Picturebook

Spacetime Processes (QM ☛ QFT)

- The un-observable exchange photon is thus virtual!
- Indeed, the two 3-particle processes ***separately*** …



- … are kinematically forbidden.
- Q.: Why?
- A.: Conservation of 4-momentum.

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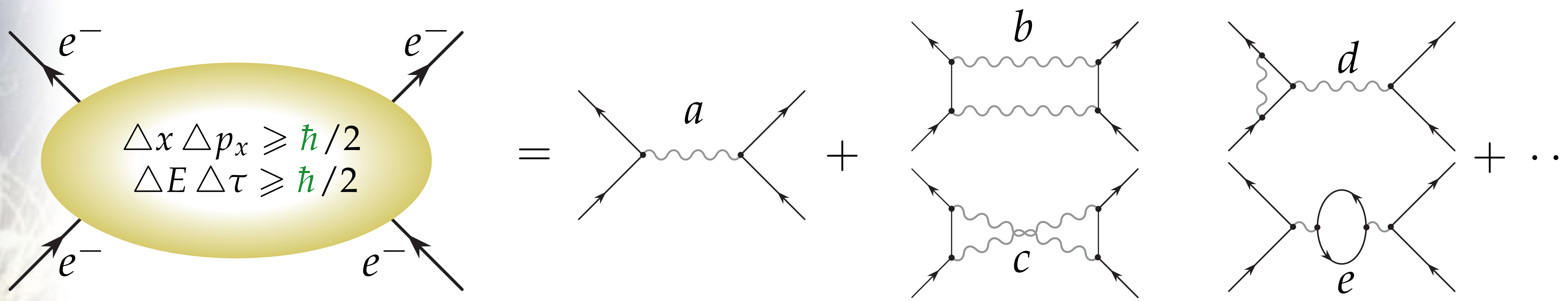
Spacetime Processes (QM ☛ QFT)

- The "process" is whatever takes "input" ⟶ "output":



- *Whatever is not forbidden, is mandatory. –R. Feynman*
- Notice, these sub-processes can be ordered:
  - by counting the "elementary" interaction vertices
  - by counting loops
  - by (non-)planarity
- …which are qualitative—**<u>& depicted</u>**—characteristics

Perturbation Picturebook

Spacetime Processes (QFT)

- A 1–1 correspondence:
  - the fundamental theory that designs the considered process,
  - diagram elements depicting terms from a Lagrangian,
  - rules of linking graphical elements, depicting a computation with the individual terms from the specified Lagrangian,
  - rules of listing all possible—and needed—Feynman diagrams,
  - the final mathematical expression for the matrix element (amplitude of probability) for the considered process, as a weighted sum of sub-processes,
  - the computation (estimate) of this mathematical expression.
- This would be the goal of a Quantum Field Theory course…
- …which this is not. ☺

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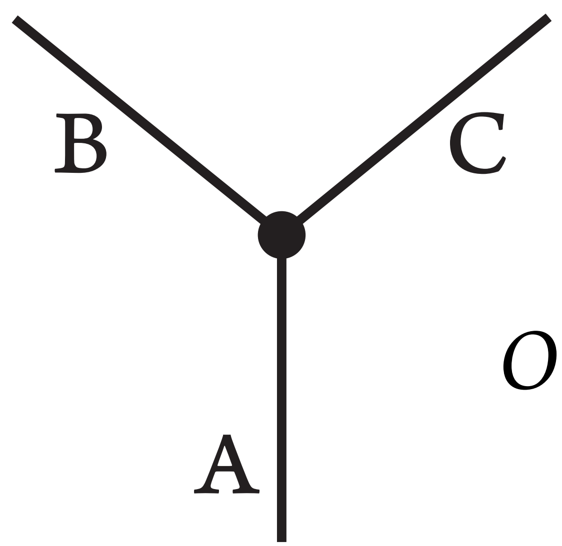
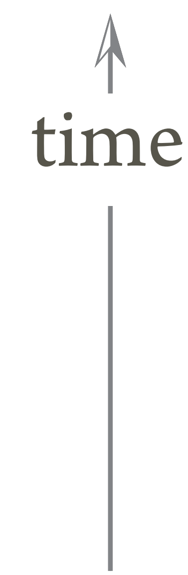
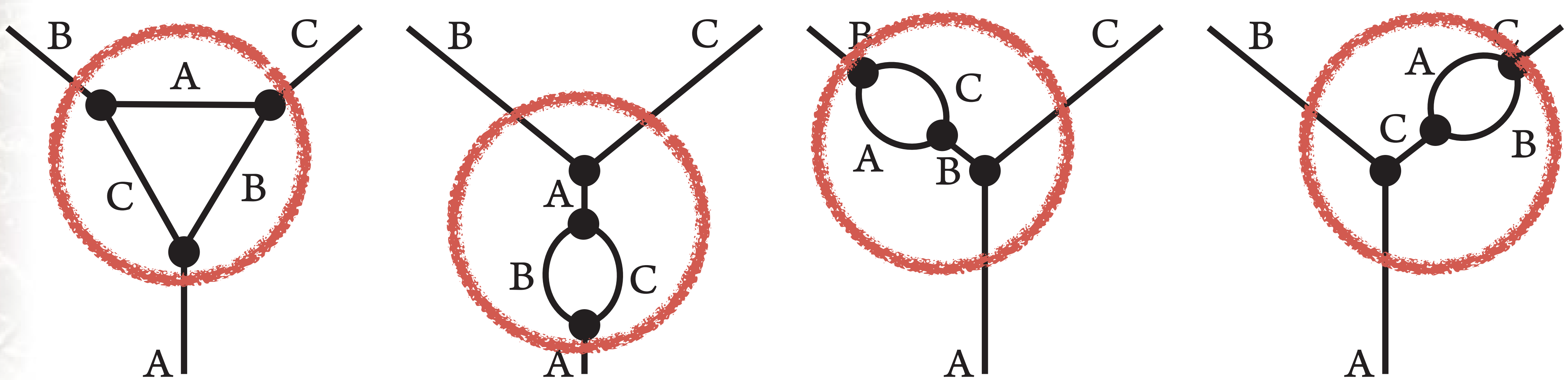
Spacetime Processes (QFT)

- Back in the *A*-*B*-*C* toy model, the $A \longrightarrow B+C$ decay.





$O(g^1)$ no loops



$O(g^3)$ one loop

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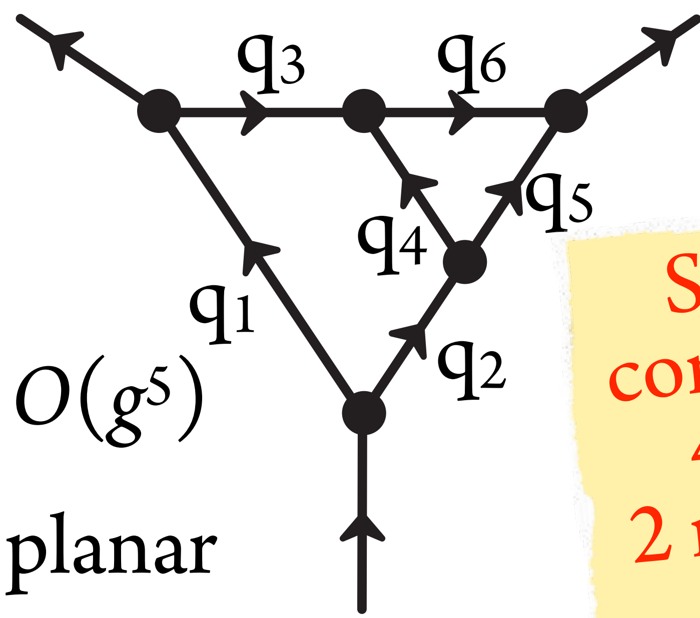
Spacetime Processes (QFT)

- And from $O(g^5)$ onward, there are also complications:

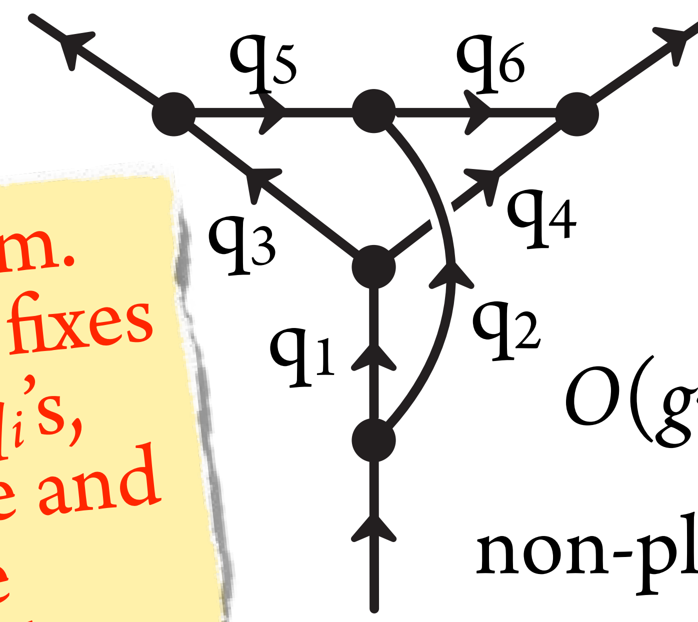


$O(g^5)$

planar

two loops

Since 4-mom. conservation fixes 4 of the 6 q_i's, 2 remain free and must be integrated!



$O(g^5)$

non-planar

# of loops ?

```
Solve[{z1 == q1 + q2, q1 == z2 + q3,
  q2 == q4 + q5, q4 + q3 == q6,
  q5 + q6 == z3, z1 == z2 + z3}]

{q2 → -q1 + z2 + z3,
 q4 → -q1 + q6 + z2, q5 → -q6 + z3,
 q3 → q1 - z2, z1 → z2 + z3}}
```

```
Solve[{z1 == q1 + q2, q1 == q3 + q4,
  q3 == z2 + q5, q4 + q6 == z3,
  q2 + q5 == q6, z1 == z2 + z3}]

{q1 → q5 - q6 + z2 + z3,
 q3 → q5 + z2, q4 → -q6 + z3,
 q2 → -q5 + q6, z1 → z2 + z3}}
```

A Few Stray Comments

OVERVIEW

Molecular Physics	molecules (bound states of atoms) (foundation of all chemistry!)
Atomic Physics	atoms (e^- & nuclei bound states)
Nuclear Physics	atomic nuclei (p^+ & n^0 bound states)
Elementary Particle Physics	hadrons (quark bound states)
	quarks, leptons, photons,…

Notice the extraordinary double-duty done by the physics discipline of Elementary Particle Physics—possibly including also a third tier, as professed to exists by such as string theory.

Perturbation Picturebook

Conservation Laws

- Strict Conservation Laws
  - Spacetime
    - Continuous (4-momentum & angular momentum)
    - Discrete (P, T, C)
  - Not Spacetime
    - EM charge
    - Chromodynamic Color
    - Weak Isospin
- Phenomenological Conservation Laws
  - lepton number(s, incl. e^-, μ^- & τ^- separately)
  - baryon number(s, incl. strangeness, charm, …)

Thanks!

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