(Fundamental) Physics of Elementary Particles

Relativistic kinematics (more detail), particle decays & scattering, etc.

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Fundamental Physics of Elementary Particles

Program

- Relativistic Kinematics (More Detail)
  - Space & time mixing, "usual" relativistic effects, revisited
  - Definitions and some crucial minuses
- Particle Decay
  - 2-particle decay kinematics
  - *n*-particle decay generalization
- Particle Collisions & Scattering
  - The CM-system *vs*. the target system
  - Fusing collisions, and process threshold
- Invariance & covariance *vs*. conservation
- Quantum Kinematics — The Heisenberg Zone
- Charge Conservation — Gauge & Other *Charges*

Relativistic Kinematics

Lorentz Transformations

- The space-time coordinate transformations that leave Maxwell's equations invariant *mix* space and time:

$$\vec{r}\,' = \vec{r} + (\gamma-1)(\hat{v}\cdot\vec{r})\,\hat{v} - \gamma\vec{v}t, \qquad \vec{r} = \vec{r}\,' + (\gamma-1)(\hat{v}\cdot\vec{r}\,')\,\hat{v} + \gamma\vec{v}t',$$

$$t' = \gamma\Big(t - \frac{\vec{v}\cdot\vec{r}}{c^2}\Big), \qquad t = \gamma\Big(t' + \frac{\vec{v}\cdot\vec{r}\,'}{c^2}\Big),$$

$$\gamma := \Big(1 - \frac{\vec{v}\,^2}{c^2}\Big)^{-\frac{1}{2}}, \qquad \hat{v} := \frac{\vec{v}}{\sqrt{\vec{v}\,^2}}.$$

- Or, put another way:

$$\vec{r}\,' = \vec{r}_\perp + \gamma\,(\vec{r}_\parallel - \vec{v}\,t), \qquad \vec{r} = \vec{r}\,'_\perp + \gamma\,(\vec{r}\,'_\parallel + \vec{v}\,t)$$

- …positions change in the direction of motion only.

Relativistic Kinematics

Consequences of Lorentz Transformations

- Relativity of simultaneity $(t_A = t_B)$:

$$t'_i = \gamma\Big(t_i - \frac{\vec{v}\cdot\vec{r}_i}{c^2}\Big), \quad i = A, B, \qquad \Rightarrow \qquad t'_A - t'_B = \gamma\,\frac{\vec{v}\cdot(\vec{r}_B - \vec{r}_A)}{c^2}$$

- Relativity of length/distance/extent:

$$\triangle\vec{r}\,' = \triangle\vec{r} + (\gamma-1)(\hat{v}\cdot\triangle\vec{r})\hat{v} = \triangle\vec{r}_\perp + \gamma\triangle\vec{r}_\parallel,$$

FitzGerald-Lorentz contraction

$$\triangle\vec{r}\,'_\parallel = \gamma\,\triangle\vec{r}_\parallel, \qquad \triangle\vec{r}\,'_\perp = \triangle\vec{r}_\perp,$$

- Relativity of duration/passage of time:

$$t_B - t_A = \gamma(t'_B - t'_A) + \gamma\frac{\vec{v}\cdot(\vec{r}\,'_B - \vec{r}\,'_A)}{c^2}.$$

time dilation

$$\triangle t = \gamma\,\triangle t'$$

- Relativity of … well, relative velocities:

$$\vec{u} := \frac{\triangle\vec{r}}{\triangle t} = \frac{\vec{u}\,'_\parallel + \vec{v}}{\big(1 + \frac{(\vec{v}\cdot\vec{u}\,')}{c^2}\big)} + \frac{\vec{u}\,'_\perp}{\gamma\big(1 + \frac{(\vec{v}\cdot\vec{u}\,')}{c^2}\big)},$$

addition of velocities

$$\vec{u}\,'_\parallel = (\vec{u}\,'_\parallel\cdot\hat{v})\hat{v}, \qquad \vec{u}\,'_\perp\cdot\hat{v} = 0.$$

Relativistic Kinematics

Conventions and Details

- Frequently useful expansions:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \approx 1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + \frac{5}{16}\beta^6 + O(\beta^8), \qquad \beta := \frac{v^2}{c^2} \ll 1;$$

$$\text{and} \qquad \approx \frac{1}{\sqrt{2\epsilon}}\Big[1 + \frac{1}{4}\epsilon + \frac{3}{32}\epsilon^2 + \frac{5}{128}\epsilon^3 + O(\epsilon^4)\Big]. \qquad \epsilon := \Big(1 - \frac{|\vec{v}|}{c}\Big) \ll 1.$$

- Notation (Cartesian coord's):

$$\mathrm{x} := \sum_{\mu=0}^{3} x^\mu\, \hat{\mathrm{e}}_\mu, \qquad \text{where} \qquad x^0 = ct, \;\; \vec{r} = \sum_{i=1}^{3} x^i \hat{\mathrm{e}}_i,$$

- … so Lorentz transformations are linear:

$$y^\mu = \mathrm{L}^\mu{}_\nu\, x^\nu, \quad \Leftrightarrow \quad \mathrm{y} = \mathbf{L}\,\mathrm{x} \quad \Leftrightarrow \quad \begin{bmatrix} y^0 \\ y^1 \\ y^2 \\ y^3 \end{bmatrix} = \begin{bmatrix} \mathrm{L}^0{}_0 & \mathrm{L}^0{}_1 & \mathrm{L}^0{}_2 & \mathrm{L}^0{}_3 \\ \mathrm{L}^1{}_0 & \mathrm{L}^1{}_1 & \mathrm{L}^1{}_2 & \mathrm{L}^1{}_3 \\ \mathrm{L}^2{}_0 & \mathrm{L}^2{}_1 & \mathrm{L}^2{}_2 & \mathrm{L}^2{}_3 \\ \mathrm{L}^3{}_0 & \mathrm{L}^3{}_1 & \mathrm{L}^3{}_2 & \mathrm{L}^3{}_3 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}.$$

Relativistic Kinematics

Conventions and Details

- Lorentz boosts:

$$\mathbf{L} = \begin{bmatrix} \gamma & -\gamma\frac{v_x}{c} & -\gamma\frac{v_y}{c} & -\gamma\frac{v_z}{c} \\ -\gamma\frac{v_x}{c} & 1+(\gamma-1)\frac{v_x^{\,2}}{\vec{v}^{\,2}} & (\gamma-1)\frac{v_x v_y}{\vec{v}^{\,2}} & (\gamma-1)\frac{v_x v_z}{\vec{v}^{\,2}} \\ -\gamma\frac{v_y}{c} & (\gamma-1)\frac{v_y v_x}{\vec{v}^{\,2}} & 1+(\gamma-1)\frac{v_y^{\,2}}{\vec{v}^{\,2}} & (\gamma-1)\frac{v_y v_z}{\vec{v}^{\,2}} \\ -\gamma\frac{v_z}{c} & (\gamma-1)\frac{v_z v_x}{\vec{v}^{\,2}} & (\gamma-1)\frac{v_z v_z}{\vec{v}^{\,2}} & 1+(\gamma-1)\frac{v_z^{\,2}}{\vec{v}^{\,2}} \end{bmatrix}$$

$$\frac{\partial \mathrm{L}^{\mu}{}_{\nu}}{\partial x^{\rho}} = 0, \quad (\mu,\nu,\rho = 0,1,2,3), \qquad \mathbf{L}^{T}\boldsymbol{\eta} = \boldsymbol{\eta}\,\mathbf{L}^{-1}, \qquad \det(\mathbf{L}) = 1,$$

- leave invariant the "proper time," τ:

$$c^2\,\tau^2 = \mathrm{x}^2 = \mathrm{x}\cdot\mathrm{x} := x^{\mu}\,\eta_{\mu\nu}\,x^{\nu}$$

- where $[\eta_{\mu\nu}] = \mathrm{diag}[1,-1,-1,-1]$ is the ***spacetime*** metric.
- $s = -\,c\tau$ is the "interval," $\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2$ the "line element."

Relativistic Kinematics

Conventions and Details

- The oft-cited special case:

$$[\mathrm{L}^{\mu}{}_{\nu}] = \begin{bmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{aligned} v &=: c\tanh(\phi), \\ \gamma &= \cosh(\phi), \\ \tfrac{v}{c}\gamma &= \sinh(\phi); \end{aligned}$$

- is related (by analytic continuation) to a rotation:

$$\begin{bmatrix} c\,(it') \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 & 0 \\ \sin(\phi) & \cos(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\,(it) \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

- and the coordinates $(x^0 = cit, x^1, x^2, x^3)$ span the "World of (Hermann) Minkowski;" but $t \to it$ is "Wick-rotation."

Relativistic Kinematics

Energy and Momentum

- (Hamiltonian action) ∝ ("world line" length):

$$S = -\int_A^B \mathrm{d}(c\tau)\,\alpha \overset{(1.8)}{=} -\int_{t_A}^{t_B} \mathrm{d}t\,\frac{\alpha c}{\gamma},$$

$$L = -\alpha c\sqrt{1-\frac{v^2}{c^2}} \approx -\alpha c + \frac{1}{2}\alpha c\frac{v^2}{c^2} + \alpha c\,O\Big(\frac{v^4}{c^4}\Big)$$

- … where we set $\alpha = mc$, leading to:

$$= -mc^2\gamma^{-1} \;=\; -mc^2\sqrt{1-\frac{\vec{v}^{\,2}}{c^2}} \;=\; -mc^2\sqrt{1-\frac{1}{c^2}|\dot{\vec{r}}|^2}.$$

- Thereupon:

$$\vec{p} := \frac{\partial L}{\partial \dot{\vec{r}}} = \frac{\partial L}{\partial \vec{v}} = m\gamma\vec{v}$$

$$E := \vec{p}\cdot\dot{\vec{r}} - L = m\gamma\,\vec{v}\cdot\vec{v} + mc^2\gamma^{-1} = m\gamma c^2$$

Relativistic Kinematics

Energy-Momentum

- Note: $E = \gamma m c^2$ is ***total energy***; $E_0 = mc^2$ is ***rest energy***.
- Also: $T = E - E_0 = m(\gamma - 1)c^2$ is ***kinetic energy***.
- The energy-momentum 4-vector (4-momentum) is

$$\mathrm{p} = (p_\mu) := (-E/c, \vec{p}) \ = (-m\gamma c, m\gamma \vec{v})$$

- the Lorentz-invariant square of which is:

$$\mathrm{p}^2 := p_\mu \, \eta^{\mu\nu} p_\nu = E^2/c^2 - \vec{p}^{\,2} = m^2\gamma^2 c^2 - \vec{p}^{\,2}$$

$$= m^2\gamma^2 c^2 \left(1 - \frac{v^2}{c^2}\right) = m^2 c^2.$$

- This defines the Lorentz-invariant mass:

$$m^2 c^4 = \mathrm{p}\cdot\mathrm{p} = E^2 - \vec{p}^{\,2} c^2 \qquad (\text{just like } c^2\tau^2 = \mathrm{x}\cdot\mathrm{x} = c^2 t^2 - \vec{r}^{\,2})$$

- and the Lorentz-invariant combination of E, p_x, p_y & p_z.

Relativistic Kinematics

Conventions and Details

- Now about that sign in $\mathrm{p} = (-E/c, \vec{p})$:
- Starting from quantum theory in coord. representation,

$$p_\mu = \frac{\hbar}{i}\frac{\partial}{\partial x^\mu}$$

- we identify:

$$p_0 = \frac{\hbar}{i}\frac{\partial}{\partial x^0} = \frac{\hbar}{i}\frac{\partial}{\partial (ct)} = -\frac{1}{c}\, i\hbar\frac{\partial}{\partial t} = -\frac{1}{c}H, \qquad \vec{p} = +\frac{\hbar}{i}\vec{\nabla}.$$

- The same may be derived by purely classical arguments:

$$(v^\mu) := \frac{\partial x^\mu}{\partial t} = (c, \dot{x}^1, \dot{x}^2, \dot{x}^3) \qquad S = -\int_{t_A}^{t_B} \mathrm{d}t\, mc^2\sqrt{1-\frac{\vec{v}^{\,2}}{c^2}} = -\int_{x^0_A}^{x^0_B} \mathrm{d}x^0\, L_0,$$

$$L_0 := m\sqrt{c^2-\vec{v}^{\,2}}, \qquad p_0 := c\,\frac{\partial\left(-m\sqrt{c^2-\vec{v}^{\,2}}\right)}{\partial c} = -m\gamma c = -E/c,$$

$$p_\mu := \frac{\partial L_0}{\partial \frac{\partial x^\mu}{\partial x^0}} = \frac{\partial L_0}{\frac{1}{c}\partial \dot{x}^\mu} = c\,\frac{\partial L_0}{\partial v^\mu}, \qquad p_i := c\,\frac{\partial\left(-m\sqrt{c^2-\vec{v}^{\,2}}\right)}{\partial v^i} = m\gamma\,\delta_{ij}\,v^j,$$

Decays and Collisions

General Remarks

- Strictly conserved quantities
  - the sum of (observable) 4-momenta
  - the sum of (observable) angular momenta (incl. spin)
  - the sum of (observable) Noether charges (incl. EM ch.)
- Collisions can be:

Type	Kinetic Energy	Mass
Elastic	Conserved	Conserved
Fissile/Explosive	increased	decreased
Fusing/Implosive	decreased	increased

Particle Decays

2-Particle Decay

- Consider $A \longrightarrow B + C$, with $m_A \neq 0$.
- Use the A-rest frame: $\mathrm{p}_A = (-m_A c, \vec{0})$, and

$$\mathrm{p}_B = (-E_B/c, \vec{p}_B), \quad \mathrm{p}_C = (-E_C/c, \vec{p}_C)$$

- 4-momentum conservation: $\mathrm{p}_A = \mathrm{p}_B + \mathrm{p}_C$, implies that

$$-m_A c = -E_B/c - E_C/c \quad \text{and} \quad \vec{p}_B = -\vec{p}_C$$

- This is useful, but provides no relationship between the energies and the 3-momenta.
- So, consider $\mathrm{p}_A = \mathrm{p}_B + \mathrm{p}_C$ also as a 4-*vector* equation.
- This equation is (by definition) not invariant,
- but $\mathrm{p}_A^2 = (\mathrm{p}_B + \mathrm{p}_C)^2$, $\mathrm{p}_B^2 = (\mathrm{p}_A - \mathrm{p}_C)^2$ and $\mathrm{p}_C^2 = (\mathrm{p}_A - \mathrm{p}_B)^2$ are.

Particle Decays

2-Particle Decay

- So, consider $p_A^2 = (p_B + p_C)^2$:

$$p_A^2 = (p_B + p)^2 = p_B^2 + p_C^2 + 2\,p_B \cdot p_C,$$

$$\| \qquad\qquad\qquad \|$$

$$m_A^2\,c^2 \qquad m_B^2\,c^2 + m_C^2\,c^2 + 2\Big(\frac{E_B}{c}\,\frac{E_C}{c} - \vec{p}_B \cdot \vec{p}_C\Big),$$

$$\|$$

$$m_B^2\,c^2 + m_C^2\,c^2 + 2\,\frac{E_B\,E_C}{c^2} + 2\,\vec{p}_B{}^2.$$

- ... so there is a relationship between energies and 3-momenta (and masses)!
- But, it's complicated.

Particle Decays

2-Particle Decay

- Consider instead $p_B^2=(p_A-p_C)^2$:

$$p_B^2 = (p_A - p_C)^2 = p_A^2 + p_C^2 - 2\,p_A \cdot p_C,$$

$$p_B^2 = m_B^2 c^2$$

$$p_A^2 + p_C^2 - 2\,p_A \cdot p_C = m_A^2 c^2 + m_C^2 c^2 - 2\,\frac{E_A}{c}\,\frac{E_C}{c} = m_A^2 c^2 + m_C^2 c^2 - 2\,m_A\,E_C.$$

- This is immediately solved:

$$E_C = \left(\frac{m_A^2 + m_C^2 - m_B^2}{2m_A}\right)c^2$$

And $p_C^2=(p_A-p_B)^2$ similarly yields the B↔C result for E_B.

Particle Decays

2-Particle Decay

- Use the universal (***on-shell***) relativistic relationship:

$$|\vec{p}_C| = \sqrt{\frac{E_C^{\,2}}{c^2} - m_C^2\, c^2} = c\sqrt{\Big(\frac{m_A^{\,2} + m_C^{\,2} - m_B^{\,2}}{2m_A}\Big)^2 - m_C^2},$$

$$= c\,\frac{\sqrt{(m_A + m_B + m_C)(m_A - m_B + m_C)(m_A + m_B - m_C)(m_A - m_B - m_C)}}{2m_A},$$

$$= c\,\frac{\sqrt{m_A^{\,4} + m_B^{\,4} + m_C^{\,4} - 2m_A^{\,2}m_B^{\,2} - 2m_A^{\,2}m_C^{\,2} - 2m_B^{\,2}m_C^{\,2}}}{2m_A}$$

NOTICE
All of these are constants!

- …and recall:

$$\vec{p}_B = -\vec{p}_C$$

$$E_B = \Big(\frac{m_A^{\,2} + m_B^{\,2} - m_C^{\,2}}{2m_A}\Big)c^2, \quad E_C = \Big(\frac{m_A^{\,2} + m_C^{\,2} - m_B^{\,2}}{2m_A}\Big)c^2.$$

Particle Decays

2-Particle Decay

- What about $A \longrightarrow B + C$, with $m_A = 0$?
- Well, $p_A{}^2 = (p_B + p_C)^2$ produced the result:

$$m_A^2\,c^2 = m_B^2\,c^2 + m_C^2\,c^2 + 2\,\frac{E_B\,E_C}{c^2} + 2\,\vec{p}_B{}^2,$$

- …whereby $m_A = 0$ would imply that a sum of non-negative quantities vanishes…
- …which can happen only if all of them vanish ***simultaneously***.
- So, a massless particle can only decay into two massless, and stationary particles… which is a contradiction.
- This much is true ***on-shell*** (when $E^2 = \vec{p}\,^2 c^2 + m^2 c^4$).

Particle Decays

≥3-Particle Decays

- 3- and more-particle decays:

$$\boxed{\mathrm{p} = \sum_i \mathrm{p}_i,} \quad \mathrm{p} - \mathrm{p}_i = \sum_{j \neq i} \mathrm{p}_j, \quad \mathrm{p}_i = \mathrm{p} - \sum_{j \neq i} \mathrm{p}_j, \quad \mathrm{p}_i + \mathrm{p}_j = \mathrm{p} - \sum_{k \neq i,j} \mathrm{p}_k,$$

Not overdetermined, just abundant in ways to approach any case.

- … and very many others.
- Squaring them (using the rest-frame of the un-indexed "parent" particle), obtain equations such as

$$\tfrac{1}{2}\Big(m^2 - \sum_i {m_i}^2\Big)c^4 = \sum_{j>i}\big(E_i E_j - |\vec{p}_i||\vec{p}_j|c^2 \cos(\phi_{ij})\big),$$

$$\tfrac{1}{2}\Big(m^2 - m_i^2 + \sum_{j \neq i} {m_j}^2\Big)c^4 = mc^2 \sum_{j \neq i} E_j - \sum_{\substack{j<k \\ j,k \neq i}} \big(E_j E_k - |\vec{p}_j||\vec{p}_k|c^2 \cos(\phi_{jk})\big),$$

- … and so on, with: $\mathrm{p}_i \cdot \mathrm{p}_j = p_{i\mu}\eta^{\mu\nu}p_{j\nu} = \dfrac{E_i E_j}{c^2} - |\vec{p}_i||\vec{p}_j|\cos(\phi_{ij})$.

Particle Scattering

Kinematics

- Typically, $A + B \longrightarrow C_1 + C_2 + \ldots$, for which we use

$$\mathrm{p}_1 + \mathrm{p}_2 = \sum_{i>2} \mathrm{p}_i,$$

- rewritten in as many ways as convenient, then squared.
- We may use the CM system:

$$\begin{array}{lll}\mathrm{p}_1 + \mathrm{p}_2 = \Big(-\frac{E_1}{c} - \frac{E_2}{c}, \vec{0} \Big), & i.e., & \vec{p}_1 = -\vec{p}_2, \\ \quad \| & & \\ \sum_{i>2} \mathrm{p}_i = \sum_{i>2} \Big(-\frac{E_i}{c}, \vec{0} \Big), & i.e., & \sum_{i>2} \vec{p}_i = 0,\end{array}$$

- reproduces conservation of energy and 3-momenta,

Particle Scattering

Kinematics

- We may use the target-system, say, $\mathrm{p}_2 = (-m_2 c, 0, 0, 0)$:

$$\mathrm{p}'_1 + \mathrm{p}'_2 = \left(-\frac{E'_1}{c} - m_2 c\,,\, \vec{p}'_1\right), \quad i.e., \qquad \vec{p}'_2 = \vec{0},$$

$$\|$$

$$\sum_{i>2} \mathrm{p}'_i = \sum_{i>2} \left(-\frac{E_i}{c}\,,\, \vec{p}_i\right), \qquad i.e., \quad \sum_{i>2} \vec{p}'_i = \vec{p}'_1.$$

- We cannot use the 4-vectors from the CM-system and the target-system *together*, but we can use Lorentz-invariant quantities from the two systems together:

$$(\mathrm{p}_1 + \mathrm{p}_2)^2 = \Big(\sum_{i>2} \mathrm{p}_i\Big)^2 = (\mathrm{p}'_1 + \mathrm{p}'_2)^2 = \Big(\sum_{i>2} \mathrm{p}'_i\Big)^2 = \dots$$

Particle Scattering

Fusing Collision

- Consider $A + B \longrightarrow C$, with $m_B \neq 0$ and B the target.

$$\mathrm{p}_A = (-E_A/c, \vec{p}_A), \quad \mathrm{p}_B = (-m_B c, \vec{0}), \quad \mathrm{p}_C = (-E_C/c, \vec{p}_C),$$

- Conservation of 4-momentum yields:

$$\left(-\frac{E_C}{c}, \vec{p}_C\right) = \left(-\frac{E_A}{c} - m_B c, \vec{p}_A\right) \qquad \begin{aligned} E_C &= E_A + m_B c^2, \\ \vec{p}_C &= \vec{p}_A =: \vec{p}, \end{aligned}$$

- and squaring $\mathrm{p}_A + \mathrm{p}_B = \mathrm{p}_C$ yields:

$$\mathrm{p}_C{}^2 = \mathrm{p}_A{}^2 + \mathrm{p}_B{}^2 + 2\mathrm{p}_A \cdot \mathrm{p}_B,$$

$$m_C^2 c^2 = m_A^2 c^2 + m_B^2 c^2 + 2E_A m_B, \qquad \Rightarrow \quad E_A = \frac{m_C^2 - (m_A^2 + m_B^2)}{2m_B} c^2.$$

- The probe (A) must have a precisely tuned energy for it to fuse with the target (B).

Particle Scattering

Fusing Collision

- Since $E = mc^2 + T$, in particular $T_A = E_A - m_A c^2$, and

$$T_A = \frac{m_C^2 - (m_A{}^2 + m_B{}^2)}{2m_B}c^2 - m_A c^2 = \frac{m_C^2 - (m_A + m_B)^2}{2m_B}c^2$$

- is the required kinetic energy of the probe for it so fuse with the target. (A neutron to be absorbed by ^{235}U…)
- It supports the impression that the kinetic energy is making up the difference between $(m_A + m_B)$ and m_C…
- …except, it is really the difference between the squares of these quantities, "normalized" by $2m_B$.
- This result is clearly the time-reversal of the one regarding a 2-particle decay.

Particle Scattering

Particle Production Threshold

- Consider now $A + B \longrightarrow C_1 + C_2 + \ldots$
- … and assume that the $A + B$ collision has barely enough total energy to create the resulting particles, C_i.
- The C_i are then (almost) at rest, with no kinetic energy.

$$\mathrm{p}_i\big|_{\text{t'hold}} = \left(-m_i c\,,\,\vec{0}\,\right),$$

$$\min\left[(\mathrm{p}'_A + \mathrm{p}'_B)^2\right] = \Big(\sum_i \mathrm{p}_i\big|_{\text{t'hold}}\Big)^2,$$

$$\min\left[m_A{}^2 c^2 + m_B{}^2 c^2 + 2E'_A\, m_B\right] = \Big(\sum_i m_i c\Big)^2,$$

$$(m_A{}^2 + m_B{}^2)c^2 + 2\,\min(E'_A)\, m_B = \sum_{i,j} m_i\, m_j\, c^2.$$

The threshold: $$E'_A \geqslant \frac{1}{2\, m_B}\Big(\sum_{i,j} m_i\, m_j - (m_A{}^2 + m_B{}^2)\Big)c^2$$

Particle Scattering

Particle Production Threshold

- In terms of kinetic energy:

$$T'_A \geqslant \frac{1}{2\,m_B}\Big(\sum_{i,j} m_i\, m_j - (m_A + m_B)^2\Big)c^2$$

- So, for a collision of the type $X + X \longrightarrow 3X + X^*$ (resulting in three X's and an anti-X)
- …the test-X must hit the target X with the kinetic energy of $[(4{\cdot}4-(1+1)^2)/2]m_X c^2 = 6\, m_X c^2$!
- This is more than naively expected:
  - to create $X + X^*$, shouldn't one need to invest only $2m_X c^2$?
  - No: 3-momentum before collision is ≠0,
  - …the product $3X$+X^* cannot be at rest; that costs energy.

Particle Scattering

Particle Production Threshold

- In beam-to-beam collisions, CM-frame = lab-frame.

$$\vec{p}_A = -\vec{p}_B =: \vec{p},$$

- If the colliding particles have the same mass,

$$E_A = E_B =: E, \qquad \text{and} \qquad \mathrm{p}_A + \mathrm{p}_B = \left(2E/c, \vec{0}\right)$$

- so that

$$\min\left[(\mathrm{p}_A + \mathrm{p}_B)^2\right] = \Big(\sum_i \mathrm{p}_i\big|_{\text{t'hold}}\Big)^2,$$

$$\left(-\frac{2\min(E)}{c}, \vec{0}\right)^2 = \Big(\sum_i m_i c\Big)^2, \quad \Rightarrow \quad \min(E) = \tfrac{1}{2}\sum_i m_i c^2.$$

Particle Scattering

Particle Production Threshold

- In terms of kinetic energy:

$$\min\left(\sum T_X\right) = \left(\sum_{i=1}^{4} m_X - 2m_X\right)c^2 \overset{2X\to 3X+\overline{X}}{=} 2m_X c^2,$$

- which indeed conforms to the naive expectations.
- This is the main reason for performing beam-to-beam collisions (if possible),
- …rather than bombarding a stationary target with accelerated (energized) probes.
- Before the collision, the total 3-momentum = 0.
- After the collision, the total 3-momentum = 0,
- …so the collision products can be at rest.

Particle Scattering

Relative Kinetic Energy

- Compare the CM/lab-frame and the relative frame of, say, B being the "target":

$$(\mathrm{p}_A + \mathrm{p}_B)^2 = (\mathrm{p}'_A + \mathrm{p}'_B)^2,$$

$$\Big(\frac{E_A + E_B}{c}\Big)^2 = \Big(-\frac{E'_A}{c} - m_B c\,,\ \vec{p}_A\Big)^2.$$

- Using that $m_A = m_B = m, \quad \vec{p}_A = -\vec{p}_B$

$$4E_A^2 = 2mc^2(E'_A + mc^2), \qquad T'_A = 4T_A\Big(1 + \frac{T_A}{2mc^2}\Big),$$

T/mc^2	1	2	5	10	20	50	100	$\cdots$
T'/mc^2	6	16	70	240	880	5 200	20 400	$\cdots$

Particle Scattering

Kinematics Lessons

- Energy is ***conserved***, but ***not invariant***:
  - The total energy of colliding particles before the collision equals the total energy of all collision products.
  - The c^{-1}-multiple of energy is the 0^{th} component of the Lorentz-*variant* 4-vector of energy-momentum; it changes when boosting from one reference frame into another.
- Mass is ***invariant***, but ***not conserved***:
  - The square-root of the Lorentz-invariant $(4\text{-momentum})^2$; remains unchanged from one frame to another.
  - The sum of masses of the colliding particles need not equal the sum of masses of the collision products.
- Conservation is in *time*, which is not Lorentz-invariant.

Quantum Kinematics

The Heisenberg Zone

- As is well known (for each $i = 1, 2, 3$ separately):

$$\triangle p_i \, \triangle x^i \geqslant \tfrac{1}{2}\hbar, \qquad \triangle p_0 \, \triangle x^0 = \triangle E \, \triangle \tau \geqslant \tfrac{1}{2}\hbar.$$

- This is an inherent *indeterminacy*, not an *uncertainty*.
- Elementary consequence of non-commutativity:

$$C := -i\big[\,A\,,\,B\,\big], \qquad C^\dagger = C.$$

$$\big[\,A_0\,,\,B_0\,\big] \;:=\; \big[\,(A - \langle A \rangle)\,,\,(B - \langle B \rangle)\,\big] \;=\; iC,$$

$$0 \leqslant \langle \big|A_0 - i\omega B_0\big|^2 \rangle = \langle A_0^{\,2} \rangle - i\omega\langle \big[A_0\,,\,B_0\big] \rangle + \omega^2 \langle A_0^{\,2} \rangle,$$

$$= \Delta_A^{\,2} + \omega\langle C \rangle + \omega^2 \Delta_B^{\,2}.$$

$$\min(\omega) = -\langle C \rangle / 2\Delta_B^{\,2}, \qquad \boxed{\Delta_A \, \Delta_B \;\geqslant\; \tfrac{1}{2}\,\big|\langle [A, B] \rangle\big|}.$$

Quantum Kinematics

The Heisenberg Zone

$$\Delta_A \, \Delta_B \;\geqslant\; \tfrac{1}{2}\,|\langle [A,B] \rangle|\,.$$

- If A and B are (canonically) conjugate variables, $[A, B] \neq 0$ follows from the canonical Poisson brackets.
  - But, E and τ are not *canonically* conjugate variables!
  - In fact, τ is not the parameter of time, but the duration of the process occurring at the energy E. The parameter (coordinate) of time is not a canonical variable.
  - Similarly, in field theory, p_i and x^i are not *canonically* conjugate variables; (ct, x^1, x^2, x^3) are not canonical variables (eigenvalues of observable operators in QFT).
- Non-commutativity $\supsetneq$ canonical non-commutativity.

Quantum Kinematics

The Heisenberg Zone

- The best known example:

Constant!

$$[\,x^j\,,\,p_k\,] = i\,\delta^j_k\,\hbar \quad \Rightarrow \quad \triangle_{x^j}\,\triangle_{p_k} \geqslant \tfrac{1}{2}\delta^j_k\,\hbar .$$

- But,

Variable!

$$[\,J_j\,,\,J_k\,] = i\varepsilon_{jk}{}^{\ell}\,J_\ell \quad \Rightarrow \quad \triangle_{J_j}\,\triangle_{J_k} \geqslant \tfrac{1}{2}\varepsilon_{jk}{}^{\ell}\,\big|\langle J_\ell\rangle\big| .$$

- … so J_1 and J_2 can be measured 'simultaneously,' in states with $\langle J_3\rangle = 0$, although they do not commute.
- The indeterminacy limit is state-dependent!
- And, of course,

$$[\,J^2\,,\,J_3\,] = 0, \quad \Rightarrow \quad \triangle_{J^2}\,\triangle_{J_3} \geqslant 0.$$

Charge Conservation

Electromagnetic Charge

- Consider:

$$
\begin{array}{rcll}
\vec{\nabla}\cdot\vec{E} &=& \frac{1}{4\pi\epsilon_0}4\pi\,\rho_e, & \Rightarrow \quad \frac{\partial(\vec{\nabla}\cdot\vec{E})}{\partial t} = \frac{1}{4\pi\epsilon_0}4\pi\,\frac{\partial\rho_e}{\partial t},\\
\vec{\nabla}\times(c\vec{B}) - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} &=& \frac{1}{4\pi\epsilon_0}\frac{4\pi}{c}\vec{j}_e, & \Rightarrow \quad \underbrace{\vec{\nabla}\cdot\vec{\nabla}\times(c^2\vec{B})}_{\text{vanishes trivially}} - \frac{\partial(\vec{\nabla}\cdot\vec{E})}{\partial t} = \frac{1}{4\pi\epsilon_0}4\pi\vec{\nabla}\cdot\vec{j}_e,\\
\hline
 & & & \Rightarrow \quad 0 = \frac{\partial\rho_e}{\partial t} + \vec{\nabla}\cdot\vec{j}_e.
\end{array}
$$

- … and so:

$$\frac{\partial\rho_e}{\partial t} = -\vec{\nabla}\cdot\vec{j}_e, \quad \Rightarrow \quad \frac{\mathrm{d}Q_{e,V}}{\mathrm{d}t} = -\oint_{\partial V}\mathrm{d}^2\vec{r}\cdot\vec{j}_e,$$

- This is even simpler in Lorentz-covariant notation:

$$\partial^\mu F_{\mu\nu} = \frac{1}{4\pi\epsilon_0}\frac{4\pi}{c}\,j_{e\,\nu}, \quad \Rightarrow \quad \partial^\nu j_{e\,\nu} = 4\pi\epsilon_0\frac{c}{4\pi}\partial^\nu\partial^\mu F_{\mu\nu} \equiv 0$$

- … and follows simply from $F_{\mu\nu} = -F_{\nu\mu}$.

Remember, for later…

Charge Conservation

Charges in General

- Additive charges ↔ continuous symmetries:
  - linear momentum ↔ translation in space
  - energy ↔ translation in time
  - angular momentum ↔ rotation in space
  - electromagnetic charge ↔ see Chapter 3
  - chromodynamic color ↔ see Chapter 4
  - weak isospin ↔ see Chapter 5

} Heisenberg Zone

} No Heisenberg Zone; see later…

- Multiplicative charges ↔ discrete symmetries
  - P (parity) ↔ reflection in space
  - T ↔ reversal of time
  - C ↔ Charge (Hermitian/Dirac) conjugation

Thanks!

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