(Fundamental) Physics of Elementary Particles

Introduction to the quark model, differential & Feynman's diagrammatic calculus

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Fundamental Physics of Elementary Particles

Program

- Introduction to the quark model
  - History's Lessons
  - Naive-pictoresque classification of hadrons
  - Bound states in the quark-model
- Relativistic Kinematics
  - Conservation laws
  - 4-dimensional notation & tensor calculus
  - Relativistic kinematics
- Feynman diagrams
  - Decay constant & the process amplitude
  - A toy-model

History's Lessons

Mesons and the μ-"meson"

- Yukawa (1934): a "meson" with $m \sim 150\ \mathrm{MeV}/c^2$
  - 1937: a particle w/ $m \sim 100\ \mathrm{MeV}/c^2$
  - 1946: the $100\ \mathrm{MeV}/c^2$-particle doesn't interact strongly
  - 1947: $m_\pi \sim 150\ \mathrm{MeV}/c^2$ & $m_\mu \sim 100\ \mathrm{MeV}/c^2$
  - 1947: $K^0 \to \pi^+ + \pi^-$, '49: $K^+ \to \pi^+ + \pi^+ + \pi^-$, '50: $\Lambda^0 \to p^+ + \pi^-$
- Strange particles: produced fast ($\sim 10^{-23}$ s) & in pairs, but decay slowly ($\sim 10^{-10}$–10^{-8} s)
- Strangeness, preserved by strong interactions, but violated by weak interactions.
- And the μ-"meson"? Well… it isn't a hadron! It's a (heavier) copy of the electron.

History's Lessons

The μ-on and Neutrini

- 1947, C. Powell: the difference between π and μ:

$$\pi^- \longrightarrow \mu^- + \overline{\nu}_\mu, \quad \mu^- \searrow e^- + \overline{\nu}_e + \nu_\mu.$$

- 1956, F. Reines & C. Cowan (1st big "waiting" exp.):

$$\overline{\nu}_e + p^+ \longrightarrow n^0 + e^+.$$

- 1959, R. Davis, Jr. & D.S. Harmer:

$$\overline{\nu}_e + n^0 \longrightarrow p^+ + e^-$$

- doesn't happen, confirming the lepton conservation number of Konopinski & Mahmoud (1953).

History's Lessons

The μ-on and Neutrini

- Finally,

$$\mu^- \xrightarrow{?} e^- + 2\gamma,$$

- never happens either. This implies a separate conservation law for (e^-, ν_e) and for (μ^-, ν_μ)!
- The μ-"doublet" seems utterly unnecessary

	Group	nuclear interactions	spin	number
	Leptons	only weak	half-integral	conserved
Hadrons	**Mesons**	both strong and weak	integral	not conserved
	Baryons	both strong and weak	half-integral	conserved*

*Baryon number (albeit not under that name) conservation was already in 1938 proposed by Ernst Stückelberg, to explain the absence of the $p^+ \to e^+ + \pi^0$ proton decay.

History's Lessons

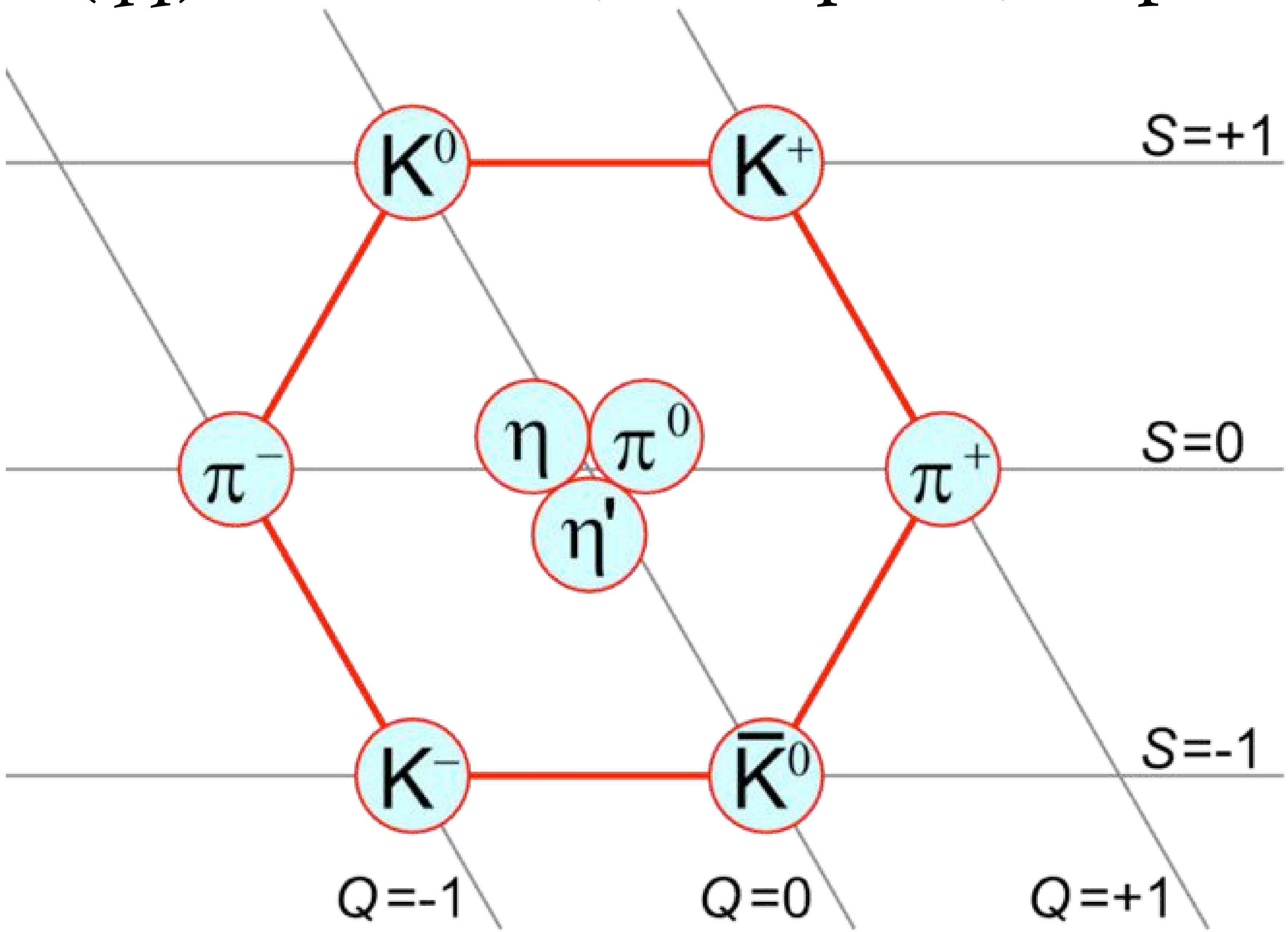
Growing Number of Hadrons

- By 1960's, too many hadrons to be elementary
- S-matrix theory
  - Initiated by Heisenberg, early 1930
  - 1960's (Geoffrey Chow, …): bootstrap model
    - We observe only *asymptotic* states, not the ~10^{-23} s hadrons
    - Hadrons consist of hadrons… …all the way down.
- Classification:
  - using isospin (Heisenberg, 1932/Wigner, 1937)
  - strangeness (Gell-Mann, 1965; Nishijima & Nakano)
  - subject to the GNN formula: $Q = I_3 + \frac{1}{2}(B + S)$

Quark Model

Quarks & mesons

- Bound $(q\bar{q})$ states with u, d & s quarks, of spin-0:



Quark Model

Quarks & mesons

- Bound $(q\bar{q})$ states of u, d & s quarks, of spin-0:

(d s*) (u s*)

(d u*) (u u*) (d d*) (s s*) (u d*)

Suitable linear combi-nations

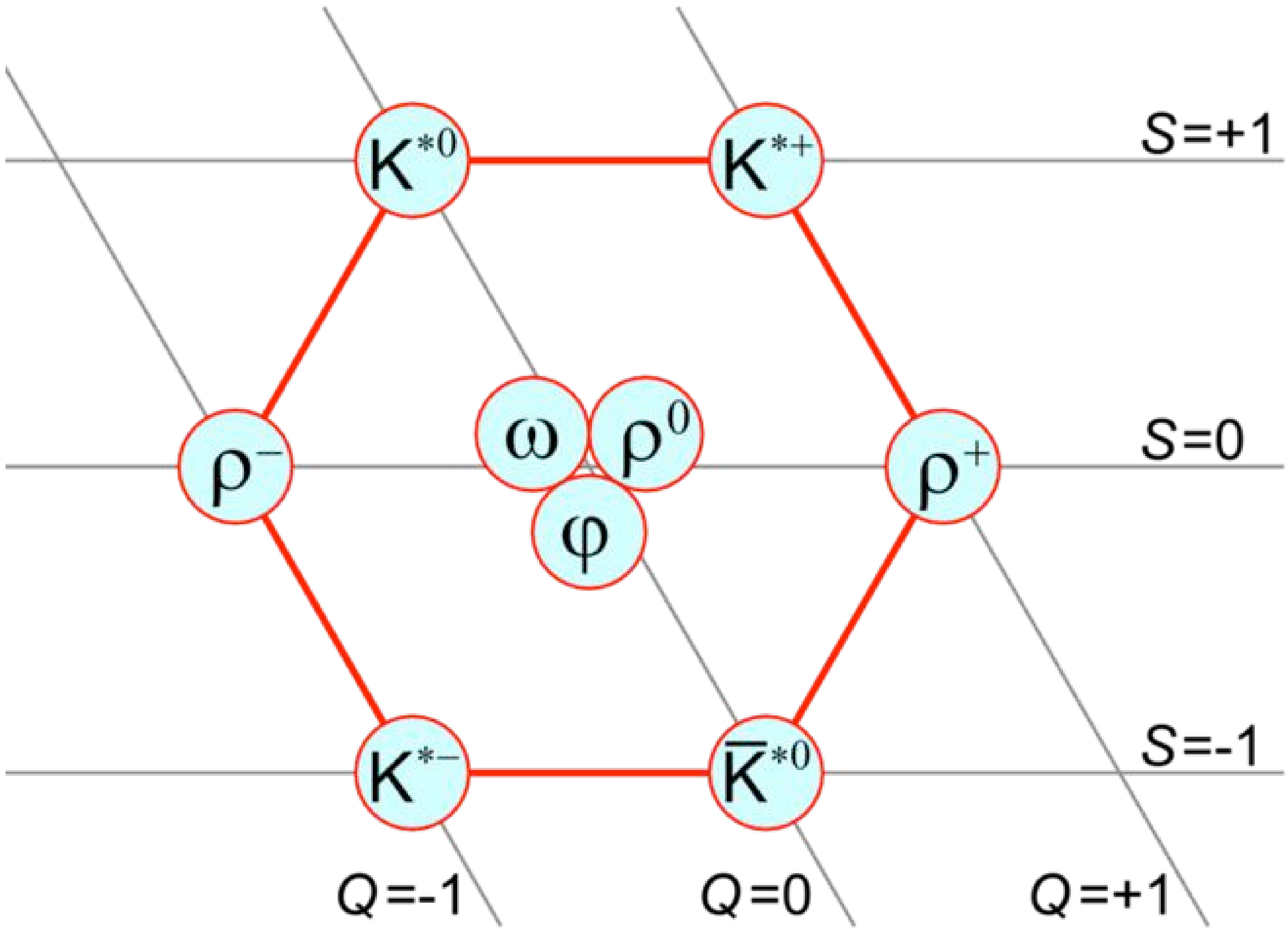
(s u*) (s d*)

Quark Model

Quarks & mesons

bound P-states
(q-orb. momentum = 1)

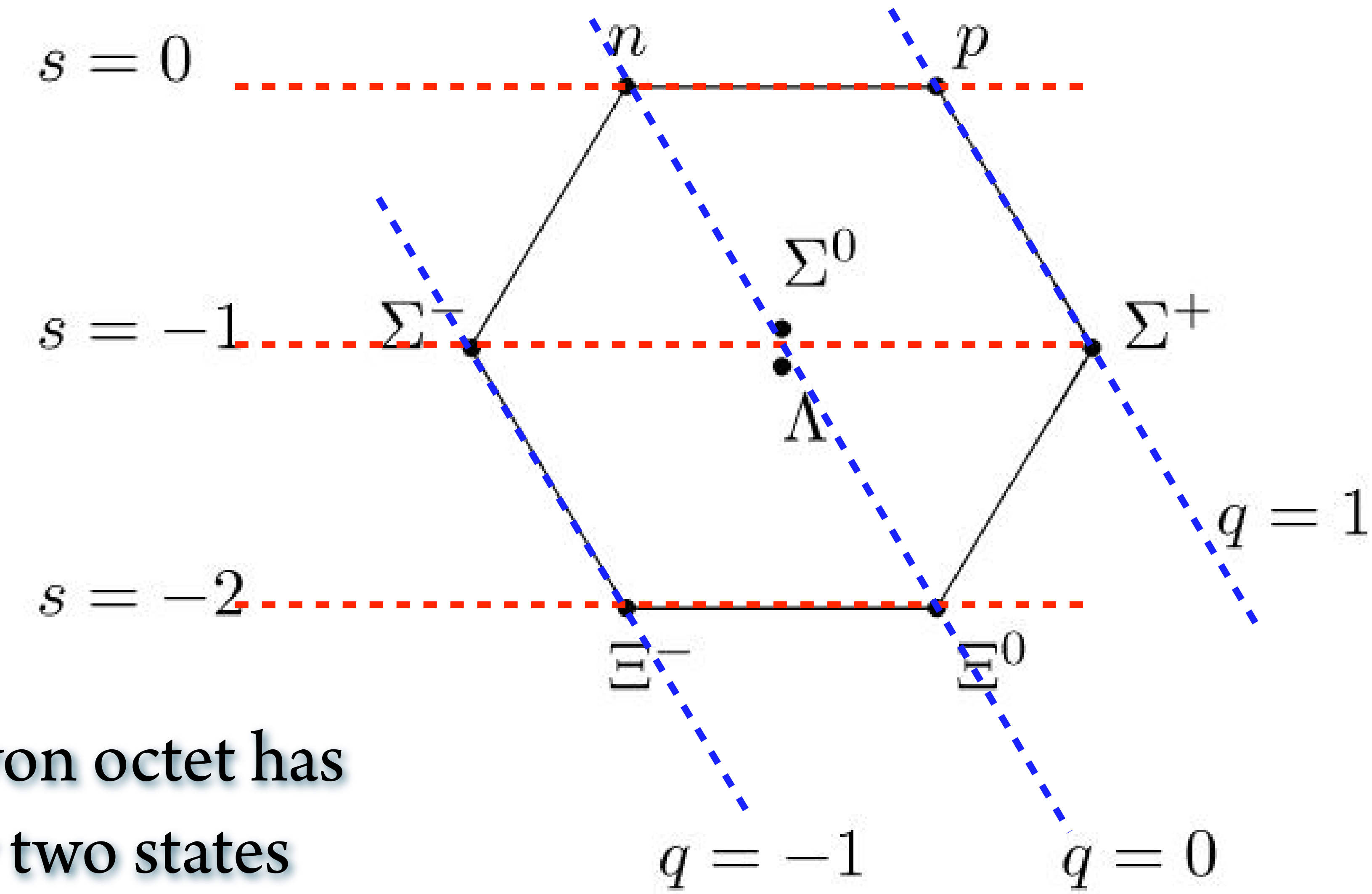
- Bound $(q\bar{q})$ states of u, d & s quarks, of spin-1:



Quark Model

Quarks & baryons

- Bound (qqq) states of u, d & s quarks, of spin-½:



Baryon octet has only two states with $s = -1$ & $q = 0$.

Quark Model

Quarks & baryons

- Bound (qqq) states of u, d & s quarks, of spin-½:

(ddu)　　(duu)

(dds)　　(dsu)　　(suu)

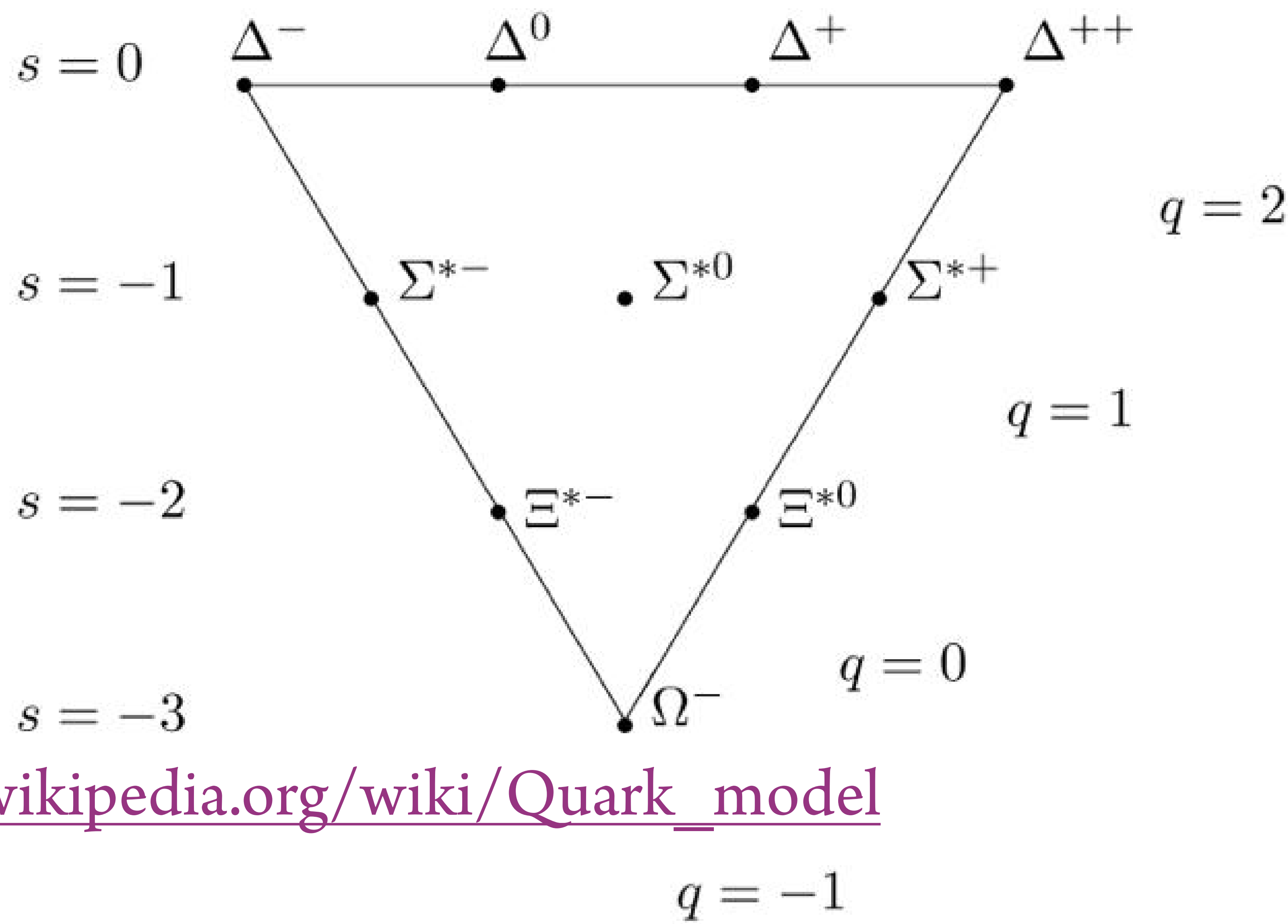
(ssd)　　(ssu)

Concrete 3-body wave-functions are quite complicated.

Quark Model

Quarks & baryons

- Bound (qqq) states of u, d & s quarks, of spin-$^3/_2$:



http://en.wikipedia.org/wiki/Quark_model

Quark Model

Quarks & baryons

- Bound (qqq) states of u, d & s quarks, of spin-$^3/_2$:

(ddd) (ddu) (duu) (uuu)

(dds) (dsu) (suu)

(ssd) (ssu)

(sss)

Concrete 3-body wave-functions are quite straightforward.

Quark Model

Quarks & baryons

- Bound state wave-functions factorize
  - $\Psi(x_1,x_2,x_3)\ \chi_1(\text{spin})\ \chi_2(\text{flavor})\ \chi_3(\text{color})$
  - Ground state: $\Psi(x_1,x_2,x_3)$ spherically-symmetric, so also w.r.t. the $x_i \leftrightarrow x_j$ exchange for every i, j pair.
  - The $\chi_3(\text{color})$ factor is antisymmetric.
  - $\Rrightarrow$ the $\chi_1(\text{spin})\ \chi_2(\text{flavor})$ factor must be symmetric w.r.t. the $i \leftrightarrow j$ exchange, for every i, j pair.
- If $\chi_1(\text{spin})$ = totally symmetric,
  - then $\chi_2(\text{flavor})$ is totally symmetric $\Rrightarrow$ decuplet

Digression: Symmetrizing

- Denote

$$|\uparrow\rangle := |\tfrac{1}{2},+\tfrac{1}{2}\rangle, \quad |\downarrow\rangle := |\tfrac{1}{2},-\tfrac{1}{2}\rangle, \quad |\uparrow\uparrow\rangle := |\tfrac{1}{2},+\tfrac{1}{2}\rangle|\tfrac{1}{2},+\tfrac{1}{2}\rangle, \quad \text{etc.}$$

- Then we have that

$$|{}^1\!/\!_2,\pm{}^1\!/\!_2\rangle\otimes|{}^1\!/\!_2,\pm{}^1\!/\!_2\rangle = \begin{cases} \begin{cases} |1,+1\rangle &= |\uparrow\uparrow\rangle, \\ |1,0\rangle &= \frac{1}{\sqrt{2}}\big[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\big], \\ |1,-1\rangle &= |\downarrow\downarrow\rangle, \qquad \text{triplet} \end{cases} \\ |0,0\rangle = \frac{1}{\sqrt{2}}\big[|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\big], \qquad \text{singlet} \end{cases}$$

- where the "triplet" is symmetric, a "singlet" is antisymmetric w.r.t. exchanging any two factors.

Digression: Symmetrizing

- The spin factor, $\chi_1(\text{spin})$, is

$$|{}^3\!/_2, m_s\rangle = \Big\{|\downarrow\downarrow\downarrow\rangle,\ |\uparrow\downarrow\downarrow\rangle_{(123)},\ |\uparrow\uparrow\downarrow\rangle_{(123)},\ |\uparrow\uparrow\uparrow\rangle\Big\},$$

  - where

$$|\uparrow\downarrow\downarrow\rangle_{(123)} := \frac{1}{\sqrt{3}}\big[|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle\big].$$

  - All linear combinations of these are totally symmetric.
- The flavor factor is

$$\chi_2(\text{flavor}) = \Big\{|ddd\rangle,\ |udd\rangle_{(123)},\ |uud\rangle_{(123)}, |uuu\rangle,\ \cdots, |sss\rangle\Big\}$$

  - ⇛ "decuplet", *i.e.*, 10 basis states.

Digression: Symmetrizing

- If the χ_1(spin) factor is mixed, for example:

$$|\tfrac{1}{2},+\tfrac{1}{2}\rangle_{[12]} = \frac{1}{\sqrt{2}}\big(|\uparrow\downarrow\uparrow\rangle - |\downarrow\uparrow\uparrow\rangle\big),\quad |\tfrac{1}{2},-\tfrac{1}{2}\rangle_{[12]} = \frac{1}{\sqrt{2}}\big(|\uparrow\downarrow\downarrow\rangle - |\downarrow\uparrow\downarrow\rangle\big)$$

  - …there exist two linearly independent states:

$$\Big\{\ |\uparrow\downarrow *\rangle_{[12]},\ \ |\uparrow * \downarrow\rangle_{[13]}\ \Big\},\quad |*\uparrow\downarrow\rangle_{[23]} = |\uparrow * \downarrow\rangle_{[13]} - |\uparrow\downarrow *\rangle_{[12]}.$$

- The product $|\uparrow\downarrow *\rangle_{[12]}|ud\star\rangle_{[12]}$ is thus symmetry w.r.t. the 1↔2 exchange, and

$$|\uparrow\downarrow *\rangle_{[12]}|ud\star\rangle_{[12]} + |\uparrow * \downarrow\rangle_{[13]}|u\star d\rangle_{[13]} + |*\uparrow\downarrow\rangle_{[23]}|\star ud\rangle_{[23]}$$

- is a totally symmetric state.

Digression: Symmetrizing

- In spin-flavor factors of the type

$$|\uparrow\downarrow *\rangle_{[12]}|ud\star\rangle_{[12]} + |\uparrow * \downarrow\rangle_{[13]}|u\star d\rangle_{[13]} + |*\uparrow\downarrow\rangle_{[23]}|\star ud\rangle_{[23]}$$

- “$*$” = ±½ i “$\star$” = u, d, s ,
- so we have spin-(±½) baryons:

(ddu) (duu)

(dds) (dsu) (suu)

(ssd) (ssu)

- where the eight three-state wave-functions are all symmetrized according to the prescription above.

Relativistic Kinematics

Relativistic kinematics

- Law: conservation of energy
- Law: conservation of momentum

} **conserv. of 4-vector energy-momentum**

- Law: conservation of angular mom. & spin-statistics
- Discrete symmetries
  - P (parity)
  - T (time reversal)
  - C (charge conjugation)
- CPT theorem
- 4-dimensional tensor calculus

Tensors in Space-Time

Diff. Calculus in Space-Time

- Derivative (general definition):
  - $D_x[f(x)\cdot g(x)] = D_x[f(x)]\cdot g(x) + f(x)\cdot D_x[g(x)]$
- Space-time: $\mathbb{R}^{1,3} := \{x^\mu \in (-\infty, +\infty),\ \mu = 0,1,2,3\}$
  - $\mathrm{x} = (x^0, x^1, x^2, x^3)$, $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$
  - $\mathrm{x}\cdot\mathrm{y} = x^0 y^0 - (x^1 y^1 + x^2 y^2 + x^3 y^3) = c^2(t_1 t_2) - \boldsymbol{r}_1\cdot\boldsymbol{r}_2$
- But — no one can force *You* to use *my* coordinates!
- Your coordinates: $y^\mu = y^\mu(x^0, x^1, x^2, x^3)$.
- … we'll need differential calculus, to compare…

Tensors in Space-Time

Diff. Calculus in Space-Time

- Differential & derivative:

$$\mathrm{d}y^\mu = \sum_{\nu=0}^{3} \frac{\partial y^\mu}{\partial x^\nu} \mathrm{d}x^\nu \quad \Longleftrightarrow \quad \frac{\partial}{\partial y^\mu} = \sum_{\nu=0}^{3} \frac{\partial x^\nu}{\partial y^\mu} \frac{\partial}{\partial x^\nu}$$

contra-variant — co-variant

tensor type (1,0) — tensor type (0,1)

- General tensor density type $(p,q;d)$:

$$\widetilde{T}^{\,\mu_1\cdots\mu_p}_{\,\nu_1\cdots\nu_p} = \left(\det\left[\frac{\partial y}{\partial x}\right]\right)^d \frac{\partial y^{\mu_1}}{\partial x^{\rho_1}} \cdots \frac{\partial y^{\mu_p}}{\partial x^{\rho_p}} \frac{\partial x^{\sigma_1}}{\partial y^{\nu_1}} \cdots \frac{\partial x^{\sigma_q}}{\partial y^{\nu_q}} T^{\,\rho_1\cdots\rho_p}_{\,\sigma_1\cdots\sigma_q}$$

Tensors in Space-Time

Diff. Calculus in Space-Time

- "Index conservation"
  - Repeated index, once up–once down, is summed; it's choice is not free, the result does not depend on it.
  - Single index is free & not summed; it's choice is free, the result does depend on it.
  - The number/position of indices in summands must agree.
  - E.g.: $A_\mu\, \eta^{\mu\nu} = A^\nu$ – "index raising"
  - $A^\mu\, \eta_{\mu\nu} = A_\nu$ – "index lowering"
  - $A_\mu\, \eta^{\mu\nu}\, A_\nu = |A_{\bullet}|^2 = A_0{}^2 - (A_1{}^2 + A_2{}^2 + A_3{}^2)$

    $= \text{square } \boldsymbol{\eta}\text{-norm of the 4-vector } (A_0, A_1, A_2, A_3)$

Tensors in Space-Time

Diff. Calculus in Space-Time

- Now, ***You*** try:
  - Tensor type of $X_\mu{}^\nu$ is $(1,1)$.
  - What is the tensor type of $X_\mu{}^\nu\, Y_\nu$, if Y_ν is of type $(0,1)$?
  - What is $X_\mu{}^\nu\, Z_{\nu\rho\sigma}$, if $Z_{\nu\rho\sigma}$ is of type $(0,3)$?
  - Is $X_{\mu\nu}\, Y_{\nu\rho\sigma}$ a tensor? – Why?
  - Is $X_{\mu\nu}\, Y^{\nu\rho\sigma}$ a tensor? – Why?
  - Is $\varepsilon^{\mu\nu\rho\sigma}\, X_{\mu\nu}\, Y_{\rho\sigma}$ a tensor, if $\varepsilon^{\mu\nu\rho\sigma}$ is a $(4,0;1)$-tensor density & $X_{\mu\nu}$ i $Y_{\rho\sigma}$ $(0,2)$-tensors? – Why?
  - Is $(\partial f/\partial x^\mu)$ a tensor? – Why?
  - Is $(\partial A^\mu/\partial x^\mu)$ a tensor? – Why?

Tensors in Space-Time

Diff. Calculus in Space-Time

- Let's check:

$$\frac{\partial f}{\partial x^\mu} \rightarrow \frac{\partial f}{\partial y^\mu} = \frac{\partial x^\nu}{\partial y^\mu}\frac{\partial f}{\partial x^\nu}$$

is a $(0,1)$-tensor. But,

$$\frac{\partial A^\mu}{\partial x^\mu} \rightarrow \frac{\partial \widetilde{A}^\mu}{\partial y^\mu} = \frac{\partial}{\partial y^\mu}\widetilde{A}^\mu(y) = \frac{\partial x^\nu}{\partial y^\mu}\frac{\partial}{\partial x^\nu}\frac{\partial y^\mu}{\partial x^\rho}A^\rho(x)$$

$$= \frac{\partial x^\nu}{\partial y^\mu}\frac{\partial y^\mu}{\partial x^\rho}\frac{\partial}{\partial x^\nu}A^\rho(x) + \frac{\partial x^\nu}{\partial y^\mu}\frac{\partial^2 y^\mu}{\partial x^\nu \partial x^\rho}A^\rho(x)$$

$$= \frac{\partial A^\nu}{\partial x^\nu} + \frac{\partial x^\nu}{\partial y^\mu}\frac{\partial^2 y^\mu}{\partial x^\nu \partial x^\rho}A^\rho(x)$$

Tensors in Space-Time

Diff. Calculus in Space-Time

- If $x^\mu \to y^\mu = \Lambda^\mu{}_\nu\, x^\nu$ is a linear transformation, and the $\Lambda^\mu{}_\nu$ matrix is constant, then

$$\frac{\partial x^\nu}{\partial y^\mu}\frac{\partial^2 y^\mu}{\partial x^\nu \partial x^\rho} = \frac{\partial x^\nu}{\partial y^\mu}\frac{\partial}{\partial x^\nu}\boxed{\frac{\partial y^\mu}{\partial x^\rho}} = \frac{\partial x^\nu}{\partial y^\mu}\frac{\partial \boxed{\Lambda^\mu{}_\rho}}{\partial x^\nu} = 0$$

- For general transformations, $x^\mu \to y^\mu = \Lambda^\mu{}_\nu\, x^\nu$, the matrix elements $\Lambda^\mu{}_\nu$ are arbitrary functions of x^ν, & the 2nd, unfortunate summand does not vanish!
- The "tensoriality" of an expression depends on the (non)linearity of the transformation $x^\mu \to y^\mu = y^\mu(x)$ that we allow.

Tensors in Space-Time

Diff. Calculus in Space-Time

- If we define

$$\frac{\mathrm{D}A^{\nu}}{\mathrm{D}x^{\mu}} := \frac{\partial A^{\nu}}{\partial x^{\mu}} + \Gamma^{\nu}_{\mu\rho}A^{\rho} \qquad \frac{\mathrm{D}A_{\nu}}{\mathrm{D}x^{\mu}} := \frac{\partial A_{\nu}}{\partial x^{\mu}} - \Gamma^{\rho}_{\mu\nu}A_{\rho}$$

- Derive how "Γ" must transform w.r.t. a nonlinear change $x^{\mu} \to y^{\mu} = y^{\mu}(x)$, so that these two D-derivative would be $(1,1)$ & $(0,2)$-tensors.
- That's the covariant derivative, Γ the Christoffel symbol.
- Linear changes $x^{\mu} \to y^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$ are Lorentz transform's; compare with electrodynamics.
  The matrices $\Lambda^{\mu}{}_{\nu}$ form the $SO(1,3)$ group. — *Huh?*

Relativistic Kinematics

Relativistic kinematics

- Useful:
  - $\mathrm{p} = (-E/c\,, p_x, p_y, p_z)$
  - $\mathrm{p}\cdot\mathrm{p} = p_\mu \eta^{\mu\nu} p_\nu = p_\mu p^\mu = E^2/c^2 - \boldsymbol{p}^2 = m^2c^2 =$ invariant
  - $E = \gamma mc^2$ for a particle of mass m, $E = pc$ if $m = 0$.
    - In non-relativistic physics, mass $m = 0$ particle is poppycock!

      (That is, its momentum, kinetic & gravitational potential energy would all have to vanish!)
  - Well-known expansion for $v < \mathrm{c}$:

$$E = \gamma\, m\, c^2 = \frac{m\, c^2}{\sqrt{1 - v^2/c^2}} = m\, c^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \cdots\right)$$

Relativistic Kinematics

Relativistic kinematics

- Two mass-m snowballs fuse at velocities $\pm 0.60\ c$. What's the mass, M, of the fused snowball?
  - Since $\boldsymbol{p}_1 = -\boldsymbol{p}_2$, momentum conservation says: $\boldsymbol{p}_M = 0$.
  - Energy conservation says $E_M = E_1 + E_2$, so $M = (E_1 + E_2)/c^2$,
  - $M = 2(mc^2\gamma)/c^2 = 2m/(1-(3/5)^2)^{1/2} = 5m/2 > 2m$.
    - The "increase" $M - 2m = \tfrac{1}{2}m$ stems from kinetic energy.
- If a mass-M lump splits into two equal, mass-m parts, what's their parting speed?
  - Energy conservation: $M = 2m\gamma$, so $v = c(1-(2m/M)^2)^{1/2}$.
  - It must be that $m < \tfrac{1}{2}M$. Since $M_{\mathrm{Deut.}} = 1875.6\ \mathrm{MeV}/c^2$, and $m_p + m_n = 1877.9\ \mathrm{MeV}/c^2$, we ***need*** 2.3 MeV-a for fission.

Relativistic Kinematics

Relativistic kinematics

- A mass-M lump splits into two lumps, masses m_1 & m_2, compute the speeds v_1 i v_2.
    - Momentum conservation says: $\boldsymbol{p}_1 = -\boldsymbol{p}_2$, jer $\boldsymbol{p}_M = 0$.
    - Energy conservation says: $E_1 + E_2 = E_M$.
    - Use that $E_i^2 - \boldsymbol{p}_i^2 c^2 = m_i^2 c^4$ and $E_M = Mc^2$, since $\boldsymbol{p}_M = 0$.
    - Solve for $|\boldsymbol{p}| = |\boldsymbol{p}_1| = -|\boldsymbol{p}_2|$ and obtain

$$\boldsymbol{p} = \pm \frac{c\sqrt{M^4 + (m_1^2 - m_2^2)^2 - 2M^2(m_1^2 + m_2^2)}}{2M}$$

- If $m_2 \approx 0$, then

$$\boldsymbol{p} = \pm \frac{c(M^2 - m_1^2)}{2M}$$

Do the math.
Plus, compute the total energies, E_1 & E_2, in terms of m_1, m_2 & M only.

Feynman Diagrams

Some basic remarks

- In decays, $dN = -\Gamma\, N\, dt$, so $N(t) = N(0)e^{-\Gamma t}$.
  - Since particles decay in several ways, $\Gamma_{\text{tot}} = \Sigma_i\, \Gamma_i$.
  - Average $\tau = 1/\Gamma_{\text{tot}}$, (*branching ratio* for i^{th} mode) $= \Gamma_i/\Gamma_{\text{tot}}$.
  - "Golden rule":

$$d\Gamma = |\mathcal{M}|^2 \frac{S}{2\hbar M}\left[\prod_{i=1}^{n} \frac{d^3\boldsymbol{p}_i}{(2\pi)^3 2E_i}\right](2\pi)^4\, \delta^4(\mathrm{p}_M - \sum_{j=1}^{n} \mathrm{p}_j)$$

- $\mathcal{M}$ is the matrix element for a mass-M particle $\rightarrow n$ particles
- p_i is the i^{th} particle 4-momentum, $E_i = c(m_i^2 c^2 + \boldsymbol{p}_i^2)^{1/2}$ its energy;
- δ-function imposes 4-momentum conservation; this in the rest-system of the decaying particle, so $p_M = (Mc, \mathbf{0})$;
- S is the product of statistical factors $1/j!$ for each j same particles.

Feynman Diagrams

Some basic remarks

- When a particle decays into two, if the 4-moments of the decay result are unknown, integrate over them:

$$\Gamma = \frac{S}{2\hbar M}\,\frac{1}{(4\pi)^2}\int \frac{\mathrm{d}^3\boldsymbol{p}_1}{\sqrt{m_1{}^2c^2+\boldsymbol{p}_1{}^2}}\,\frac{\mathrm{d}^3\boldsymbol{p}_2}{\sqrt{m_2{}^2c^2+\boldsymbol{p}_2{}^2}}\,|\mathcal{M}|^2\,\delta^4(p_M - p_1 - p_2)$$

where

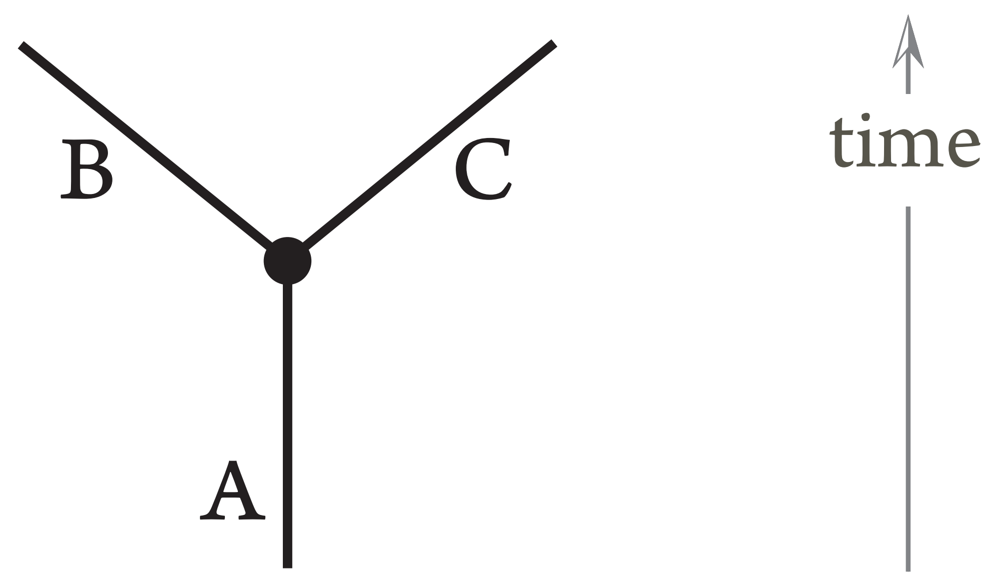
$$\begin{aligned}\delta^4(p_M - p_1 - p_2) &= \delta\big(Mc - \tfrac{E_1}{c} - \tfrac{E_2}{c}\big)\,\delta^3\big(-\boldsymbol{p}_1 - \boldsymbol{p}_2\big)\\ &= \delta\big(Mc - \sqrt{m_1{}^2c^2+\boldsymbol{p}_1{}^2} - \sqrt{m_2{}^2c^2+\boldsymbol{p}_2{}^2}\big)\,\delta^3\big(\boldsymbol{p}_1 + \boldsymbol{p}_2\big)\end{aligned}$$

  - $\boldsymbol{p}_2$-integration, b/c of δ^3-funcion, becomes $\boldsymbol{p}_2 \to -\boldsymbol{p}_1$ subst.
  - Angular $\boldsymbol{p}_1$-integration yields 4π—if $\mathcal{M}$ does not depend.
  - Typically, you are left with $|\boldsymbol{p}_1|$-integration.

Feynman Diagrams

Toy-model Feynman rules

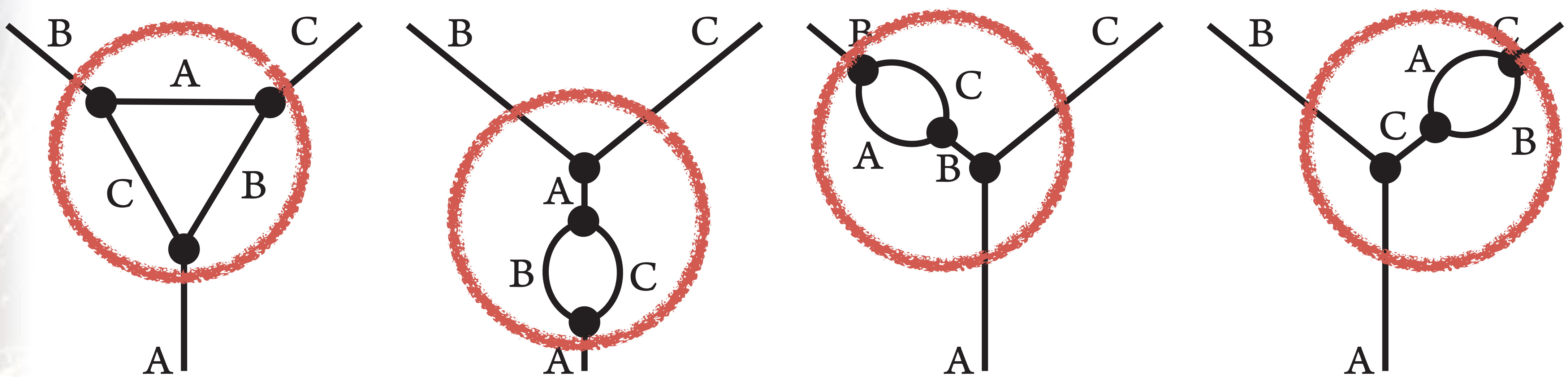
- Compute $\mathcal{M}$ using Feynman rules
- Toy-model has three particles, of mass m_A, m_B & m_C.
- Asume $m_A > m_B + m_C$, so that A → B + C is the only possible decay:



Feynman Diagrams

Toy-model Feynman rules

- Contributions to that (simplest) process:

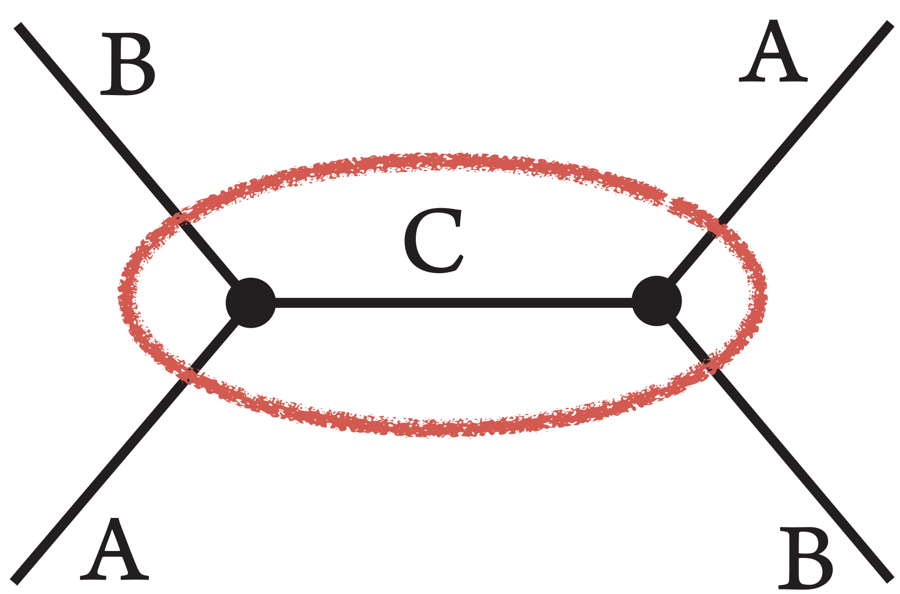


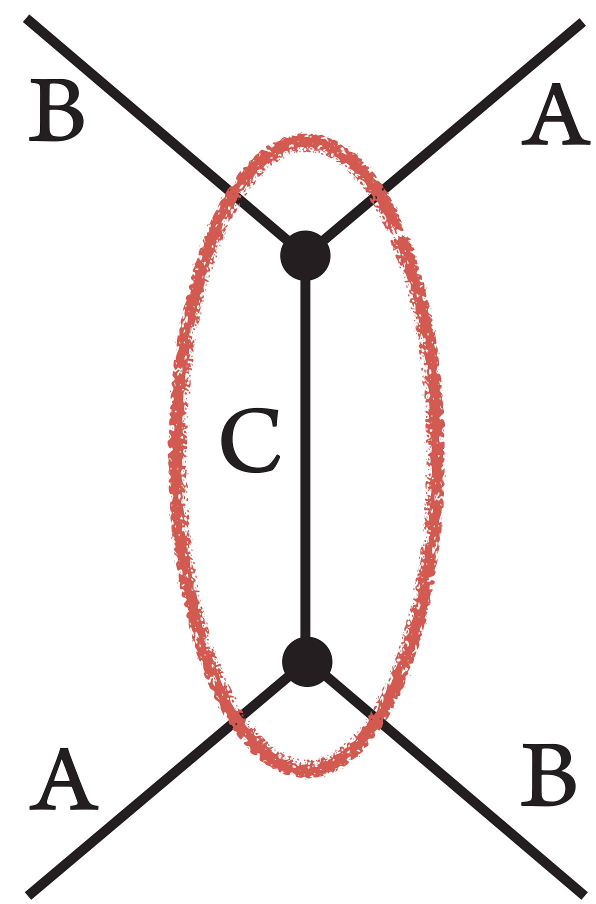
- where every "vertex" is of the A → B + C type,
- except that the $m_A > m_B + m_C$ kinematic constraint may fail
- All diagrams have 3 vertices & one closed loop; next are those with two loops & 5 vertices ...

Feynman Diagrams

Toy-model Feynman rules

- Even in the simplest case, the elastic collision can happen in ***two*** distinct ways, as depicted by (virtual histories):





What's not forbidden, is allowed.
And the rest, when no one's watching.

Feynman Diagrams

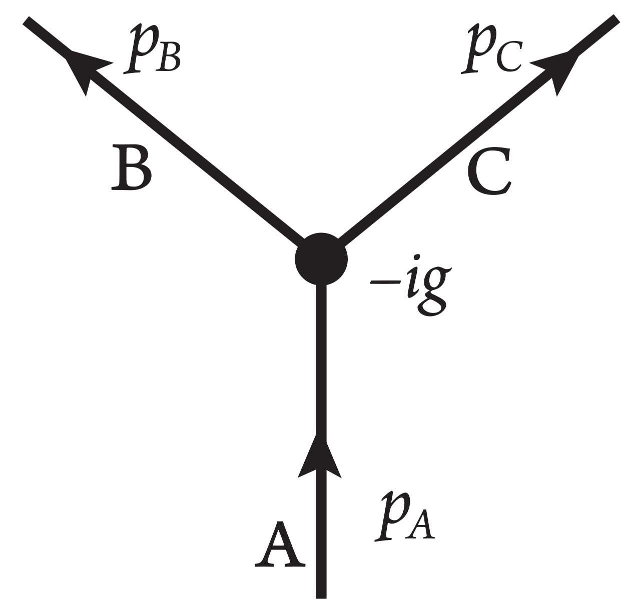
Toy-model Feynman rules

1. Denote external 4-momenta p_i, internal q_j; set the direction by orienting the lines.
2. For each vertex, write $[-i\,g]$, where g is the parameter ("strength") of the interaction.
3. For each vertex, write $(2\pi)^4\,\delta^4(k_1+k_2+k_3)$; (vertex-entering 4-momenta $= +k_i$, vertex-leaving $= -k_i$).
4. For each internal line insert a $[i/(q_j^2-m_j^2c^2)]$ factor, where the 4-momentum q_j is *free*, so $q_j^2 \neq m_j^2c^2$.
5. For each internal line insert $d^4q_j/(2\pi)^4$ & integrate.
6. From the whole mess, cancell $(2\pi)^4\,\delta^4(p_1+p_2+\ldots-p_n)$.
7. The result is $-i\mathcal{M}$.

Feynman Diagrams

Toy-model Feynman rules

- "Tree-level" value of $\Gamma_{A\to B+C}$:



- As there are no int. lines, rules 4 & 5 are void.
- Applying rules 1 & 2 we get the decorated diagram above, and rule 3 inserts the factor $(2\pi)^4\,\delta^4(p_A - p_B - p_C)$, which then rule 6 throws out.
- What remains is $\mathcal{M} = g$, in this, simplest, approximation.

Feynman Diagrams

Toy-model Feynman rules

- Now compute:

$$\Gamma = \frac{S}{2\hbar m_A} \int |\mathcal{M}|^2 \,(2\pi)^4 \delta^4(\mathrm{p}_A - \mathrm{p}_B - \mathrm{p}_C)\, \frac{c\,\mathrm{d}^3\vec{p}_B}{2(2\pi)^3 E_B(\vec{p}_B)} \frac{c\,\mathrm{d}^3\vec{p}_C}{2(2\pi)^3 E_C(\vec{p}_C)},$$

$$= \frac{S}{2(4\pi)^2\hbar m_A} \int \mathrm{d}^3\vec{p}_B\, |\mathcal{M}|^2 \frac{\delta\left(m_A c - \sqrt{m_B^2 c^2 + \vec{p}_B^2} - \sqrt{m_C^2 c^2 + (-\vec{p}_B)^2}\right)}{\sqrt{m_B^2 c^2 + \vec{p}_B^2}\sqrt{m_C^2 c^2 + (-\vec{p}_B)^2}},$$

$$= \frac{S}{8\pi\hbar m_A} \int_0^\infty \frac{\rho^2 \mathrm{d}\rho\, |\mathcal{M}|^2}{\sqrt{m_B^2 c^2 + \rho^2}\sqrt{m_C^2 c^2 + \rho^2}}\, \delta\left(m_A c - \sqrt{m_B^2 c^2 + \rho^2} - \sqrt{m_C^2 c^2 + \rho^2}\right).$$

- Substitute

$$\mathcal{E} = c\left(\sqrt{m_B^2 c^2 + \rho^2} + \sqrt{m_C^2 c^2 + \rho^2}\right), \qquad \frac{\mathrm{d}\mathcal{E}}{\mathcal{E}} = \frac{\rho(\mathcal{E})\,\mathrm{d}\rho}{\sqrt{m_B^2 c^2 + \rho^2}\sqrt{m_C^2 c^2 + \rho^2}}$$

$$\frac{\rho^2 \mathrm{d}\rho}{\sqrt{m_B^2 c^2 + \rho^2}\sqrt{m_C^2 c^2 + \rho^2}} = \rho(\mathcal{E})\frac{\mathrm{d}\mathcal{E}}{\mathcal{E}}$$

Feynman Diagrams

Toy-model Feynman rules

- and obtain:

$$\Gamma = \frac{S}{8\pi\hbar m_A} \int_{(m_B+m_C)c^2}^{\infty} \frac{\mathrm{d}\mathcal{E}}{\mathcal{E}} \, |\mathcal{M}|^2 \rho(\mathcal{E}) \, \delta(m_A c - \mathcal{E}/c),$$

$$= \begin{cases} \dfrac{S\,\rho_0}{8\pi\hbar m_A^2 c} \, |\mathcal{M}(\rho_0)|^2, & \text{if} \quad m_A > m_B + m_C; \\ 0, & \text{if} \quad m_A \leqslant m_B + m_C. \end{cases}$$

- where

$$\rho_0 = |\vec{p}_B|_0 = \frac{c}{2m_A} \sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2\, m_B^2 - 2m_A^2\, m_C^2 - 2m_B^2\, m_C^2}$$

$$\mathcal{E}(\rho_0) = m_A c^2$$

Feynman Diagrams

Toy-model Feynman rules

- For the case when $\mathcal{M} = g$, we have
  - $\Gamma = g^2 \rho_0 / 8\pi\hbar m_A{}^2 c$.
  - $\tau = 1/\Gamma = 8\pi\hbar m_A{}^2 c / g^2 \rho_0$.
- Exercises:
  - Do the math.
  - Check the units for the Γ and τ results.

Thanks!

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