1 Method of Images

This often useful and very practical technique turns out to rely on a general characteristics in physics problem solving, where we can substitute a problem[5] of interest with a less (technically) complicated one, and infer the solution for the former from that of the latter.

1.1 A Very Simple Example

Consider the sequence of sketches in Figure 1, which illustrate the *static* situation of a single,

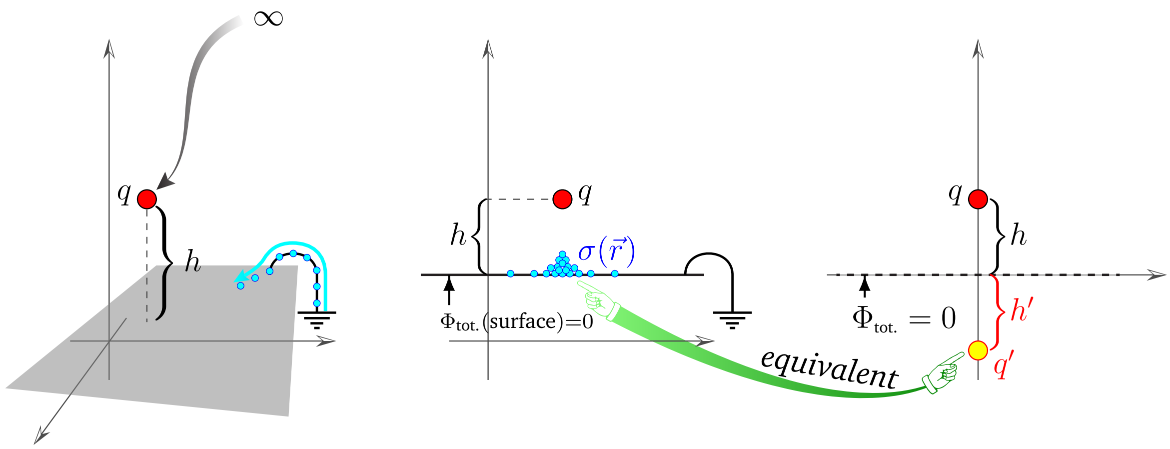


Figure 1: A positive point-charge $q > 0$, brought near a perfect, planar, grounded conductor attracts a surplus of negative charges through the grounding, and they accumulate on the surface of the conductor, under the separate point-charge. The joint scalar potential due to these surface charges has the same effect as a single "image" charge placed on the opposite side of the plane.

isolated, positive point-charge q has been brought into the vicinity of an "infinite," planar and "perfect conductor"[6] that is grounded. Although we will be concerned with this *static* situation/configuration, it will behoove us to quickly assess:

A. what may have (and even more importantly—what *must* have) transpired in the (*dynamical*) process of achieving such a configuration, and

B. what are the necessary characteristics of the final configuration.

We now discuss these two aspects in turn, and find that considerations about the former will help with the considerations about the latter.

Also, we analyze herein the situation extremely (and possibly overly) meticulously, preferring to err on the side of being (over-)pedantic and making all relevant observations explicit, including what is often left implicit and unstated in other accounts and texts [1, 2, 3, 4, 5, 6]. Note that the region of space where the brought-in separate point-charge is located evidently is physically accessible. Since the planar conductor is regarded as infinitely large—or at least (by assumption) the point-charge is not to be moved near the edges of the planar conductor where its finiteness would have observable effects—space on the other side of the planar conductor is inaccessible.

Therefore, the observable physical quantities that characterize the configuration, are:

[5] By "problem" we mean the inquiry about certain physical characteristics of a physical configuration.

[6] Certainly when discussing simple—*and idealized*—configurations, "infinite" simply means that the effects caused by its boundaries at large but finite distances are not observable. Similarly, a conductor is regarded "perfect" if the small but finite resistance to the movement of charges within the conductor can cause no observable effect.

i. the electrostatic field $\vec{E} = -\vec{\nabla}\,\Phi$ in the physically accessible region,

ii. the charge distribution within the planar conductor,

iii. the force exerted on the physical point-charge by the surface charge distribution within the planar conductor.

Subsequently, we may also compute:

iv. the work done in assembling the final configuration,

v. the work done in moving the separate point-charge a finite distance nearer to the planar conductor or farther away from it,

vi. the work done in disassembling the final configuration,

and so on; see Ref.[2, Ch. I], Ref.[3, Ex. 2.1–6], Ref.[4, § 2] and [6, § 8.2–3] for more details and calculations.

1.1.1 A Detailed and Verbose Analysis:

A. Configuration Assembly: The perfect planar conductor is initially grounded (connected by means of a "perfect" conductor to an "infinite" reservoir of movable charges) and so has no surplus charges of either sign: owing to "perfect" conductance, the movable charges are all perfectly distributed and neutralize each other.

1. The separate, point-charge, initially held at "infinity" (i.e., arbitrarily far away from the grounded conductor), has its static Coulomb field, $\vec{E}_P = -\vec{\nabla}\,\Phi_P$, where the subscript stands for "physical."
2. This field is spherically symmetric around the point-charge at the center, with $\vec{E}_P$ oriented radially outward (for $q > 0$) from the point-charge, the intensity (magnitude) of which is proportional to the reciprocal of the square of the distance from the point-charge. Consequently, Φ_P is also spherically symmetric and is proportional to the reciprocal of the distance from the point-charge.
3. As the separate, point-charge is being brought (from "infinity," to the vicinity of the grounded, perfect planar conductor, its electric field induces the movable charges within the planar conductor, the grounding wire and the ground to move: negative charges are attracted and move closer to the location where the separate point-charge is at any particular moment, positive charges are repelled and move further from that location.
4. As the separate, point-charge is brought to rest (and remains static) at a finite perpendicular distance h from the surface of the planar conductor, the movable charges within the conductor-grounding wire-ground system settle—after sufficient (*transient*) time—into a static configuration.
5. The so-assembled and re-distributed charges within the grounded planar conductor will be called "surplus," in reference to the initial, homogeneously neutral charge distribution of the planar, grounded perfect conductor before the separate, point-charge had been brought in to attract them.

The final charge distribution on the grounded planar conductor is thus described as the superposition of the initial (homogeneously neutral) charge distribution and the additional distribution of the "surplus" charges brought in by attraction to the separate, point-charge.

B. Configuration Characteristics: Several facts about the final surplus charge distribution within the planar, grounded conductor should be clear:

1. The surplus charges must be negative: Being attracted into the conductor by the separate point-charge q, they must be of opposite sign from it.
2. The surplus charges can only be on the surface of the conductor, owing to their mutual repulsion and no resistance to motion within the perfect conductor: within the conductor, they can go as far away from each other as possible so that none will stay in the inside region of the conductor.
3. Their (surplus) surface charges are densest where they are closest to the separate point-charge, and their density diminishes further out. This surface charge density distribution is determined by the balance of the two opposing types of electrostatic forces: the attractive force from the separate point-charge, and the mutual repulsive forces between the (surplus) surface charges themselves.

   This balancing requirement *could* be used to compute the surface charge density, but we will find an easier method below.
4. The (surplus) surface charge density is rotationally symmetric about the axis perpendicular to the plane conductor and passing through the separate point-charge. This follows from the fact that the electrostatic field of the separate point-charge is spherically symmetric and so certainly is also symmetric about the indicated axis; see the left-most illustration in Figure 2.

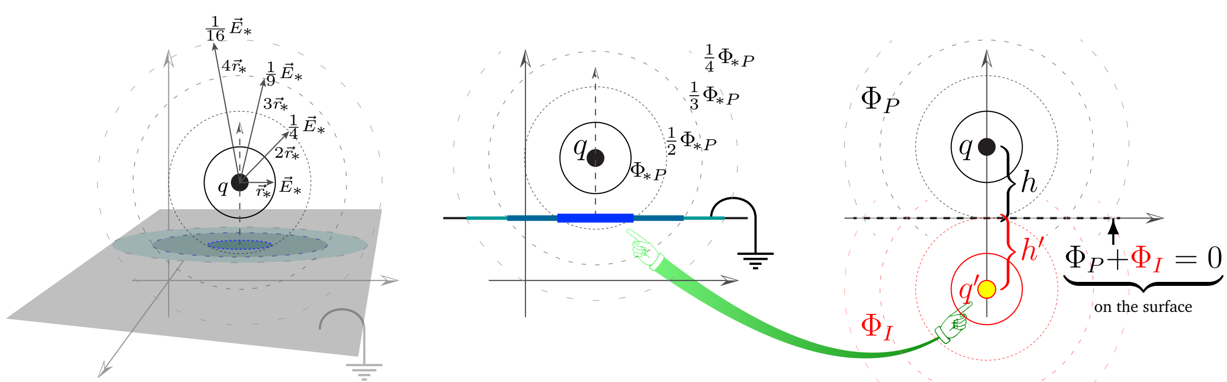


Figure 2: The electrostatic field of the positive point-charge $q > 0$ is spherically symmetric about the point-charge. It is then also axially symmetric about any axis that passes through that point-charge, and so also about the axis through the point-charge that is perpendicular to the plane conductor. The magnitude of the field diminishes with (the square of) the distance, and so does the density of the surplus surface charges.

5. The scalar potential due to the sum total of all (surplus) surface charges balances that from the separate point-charge, so that the scalar potential vanishes on the surface of the conductor: The fact that the surface charges are static implies that there is no horizontal $\vec{E}$-field along the surface[7], so that the horizontal (tangential to the surface) components of

[7]If $\vec{E}$ did have a component parallel to the surface of the planar conductor, it would move the surface charges along the surface, they would continue to re-distribute over the surface and so would not be static.

$\vec{E} = -\vec{\nabla}\,\Phi_{\text{tot.}}$ must vanish. This implies that $\Phi_{\text{tot.}}$ must be constant along the surface, and we choose this constant to be zero.

6. Owing to the fourth observation, we may change the coordinate system[8] so that the (x, y)-plane coincides with the planar conductor, and the positive z-axis passes through the point-charge q, which is now located at the height h above the origin; see the right-most sketchs in Figures 1 and 2.
7. From the symmetry of the sketches in the right-most sketches in Figures 1 and 2, we conclude that $q' = -q$ and $h' = h$. That is: the electrostatic field produced by the surface charges indicated in the middle sketches in Figures 1 and 2 is physically equivalent to the one produced by the (*substitute/replacement*) "image" charge $q' = -q$ placed into the physically inaccessible region, at the same distance from the surface of the planar conductor as the physical point-charge, but on the precisely opposite side.

The total scalar potential is the sum of the potential $\Phi_P(\vec{r})$ produced by the physical charge, and the potential $\Phi_\sigma(\vec{r})$ produced by the surplus surface charges—or equivalently $\Phi_I(\vec{r})$ produced by the image charge located in the physically inaccessible region. Both of these are observable in the physically accessible region so that $\Phi_I(\vec{r}) = \Phi_\sigma(\vec{r})$ in the physically accessible region. Both the functions $\Phi_P(\vec{r})$ and $\Phi_I(\vec{r})$ of course do extend (mathematically) into the physically inaccessible region, but they cannot be observed there and their value there is irrelevant. The sum $\Phi_{\text{tot.}}(\vec{r}) = \Phi_P(\vec{r}) + \Phi_I(\vec{r})$ is plotted in Figure 3.

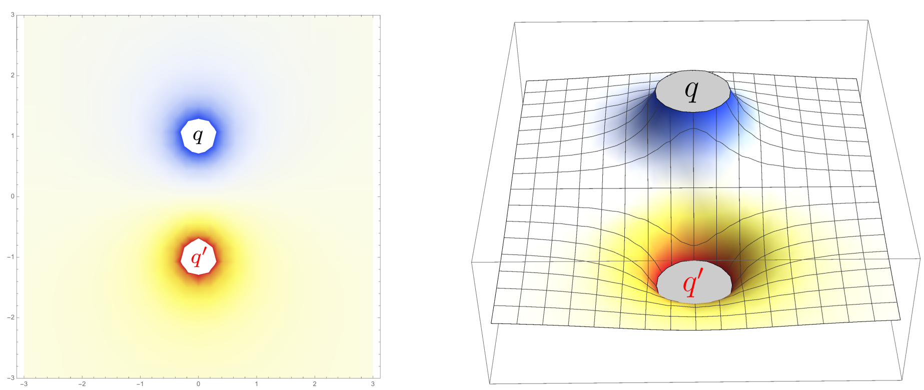


Figure 3: The scalar potential $\Phi_{\text{tot.}}(\vec{r}) + \Phi_P(\vec{r}) + \Phi_I(\vec{r})$, plotted as a function of two rectilinear coordinates: one perpendicular to the planar conductor, one parallel to it; see the sketches in display (1.1) below.

The above conclusions were all drawn (both literally in the figures and metaphorically, i.e., reasoned out) "by inspection," i.e., without any actual computation. This much is implied by Jackson's 4-line paragraph [4, bottom of p. 57] and shown in his Figure 2.1 on p. 58. The aim of these notes is to detail the thought process behind this as well as the *subsequent* computations, to aid the student in developing this insight and intuition about such problems. The latter is however acquired only through practice.

[8]We are in no way obliged to do so, and will revisit the problem also from the vantage point of a general coordinate system.

1.1.2 Computational Confirmation

We now confirm the above results by explicit computation,

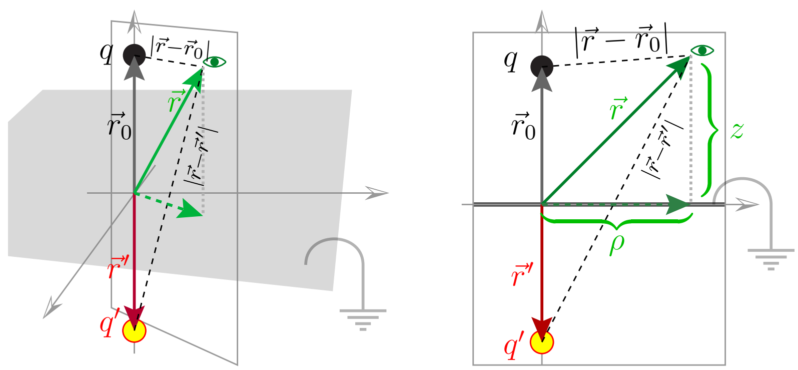
1. starting from the right-most, (substitute/replacement) "image" configuration,
2. verifying that it satisfies the boundary condition $\big[\Phi_P + \Phi_I\big]_{\text{surface}} = 0$,
3. computing the surface charge density σ such that $\Phi_\sigma = \Phi_I$.

Centered Coordinates: Choose the axis that passes through the separate, physical point-charge as the z-axis and choose the point where this z-axis passes through the planar conductor as the origin. Use cylindrical coordinates (ρ, ϕ, z), so that the surface of the planar conductor is the (ρ, ϕ)-plane; axial symmetry (rotations about the z-axis) then implies that a solution can be found that does not depend on the ϕ-angle.

The location of the physical point-charge q is at $\vec{r}_0 = (0, 0, h)$, while the image charge is located at $\vec{r}' = (0, 0, -h)$. The distance of an arbitrary point $\vec{r} = (\rho, \phi, z)$ from the locations of the two charges is then

distance from $q > 0$ is:
$$|\vec{r} - \vec{r}_0| = \sqrt{\rho^2 + (z-h)^2},$$

distance from q' is:
$$|\vec{r} - \vec{r}'| = \sqrt{\rho^2 + (z+h')^2}.$$



(1.1)

On the surface of the planar conductor, $z = 0$, and the condition $\big[\Phi_P + \Phi_I\big]_{\text{surface}} = 0$ becomes

$$0 \stackrel{!}{=} \frac{1}{4\pi\epsilon_0}\Big[\frac{q}{|\vec{r}-\vec{r}_0|} + \frac{q'}{|\vec{r}-\vec{r}'|}\Big] = \frac{1}{4\pi\epsilon_0}\Big[\frac{q}{\sqrt{\rho^2+h^2}} + \frac{q'}{\sqrt{\rho^2+(h')^2}}\Big], \qquad \text{for } \rho \in [0, \infty) \tag{1.2}$$

which must hold over the whole surface of the planar conductor, i.e., for all (ρ, ϕ); as the expressions are independent of ϕ, only the dependence on ρ matters, as indicated above. It is clear that setting $q' = -q$ and $h' = \pm h$ does satisfy this condition for all ρ:

$$0 \stackrel{!}{=} \frac{1}{4\pi\epsilon_0}\Big[\frac{q}{\sqrt{\rho^2+h^2}} + \frac{-q}{\sqrt{\rho^2+(\pm h)^2}}\Big] = \frac{1}{4\pi\epsilon_0}\Big[\frac{q}{\sqrt{\rho^2+h^2}} - \frac{q}{\sqrt{\rho^2+h^2}}\Big] \;\equiv\; 0, \quad \text{for } \rho \in [0, \infty), \tag{1.3}$$

but it may not be clear that this is the *only* solution. To see this, expand these expressions for the region where $\rho > h, h'$ by using the results (A.4) with $a = \rho^2$ and $b = h^2$ and $b = (h')^2$, respectively:

$$0 \stackrel{!}{=} \frac{1}{4\pi\epsilon_0}\Big[\frac{q}{\rho}\Big(1 - \frac{1}{2}\frac{h^2}{\rho^2} + \frac{3}{8}\frac{h^4}{\rho^4} - \frac{5}{16}\frac{h^6}{\rho^6} + \cdots\Big) + \frac{q'}{\rho}\Big(1 - \frac{1}{2}\frac{(h')^2}{\rho^2} + \frac{3}{8}\frac{(h')^4}{\rho^4} - \frac{5}{16}\frac{(h')^6}{\rho^6} + \cdots\Big)\Big], \tag{1.4}$$

$$= \frac{1}{4\pi\epsilon_0}\Big[\frac{q+q'}{\rho} - \frac{1}{2}\frac{qh^2+q'(h')^2}{\rho^2} + \frac{3}{8}\frac{qh^4+q'(h')^4}{\rho^4} - \frac{5}{16}\frac{qh^6+q'(h')^6}{\rho^6} + \cdots\Big]. \tag{1.5}$$

Since the powers of the free variable ρ are linearly independent, every one of these infinitely many terms must vanish separately, and it should be fairly obvious that this can happen only if

$$q' = -q \qquad \text{and} \qquad h' = \pm h. \tag{1.6}$$

In fact, these conditions are enforced by the vanishing of the first and the second term, respectively, after which the remaining infinitely many terms cancel automatically.

The solution $h' = -h$ would place the image charge $q' = -q$ (with $\vec{r}' = (0,0,-h')$ coordinates) at the location of the physical charge q itself, and so neutralize it. This mathematical solution of course corresponds to the trivial satisfaction of the condition $\big[\Phi_P + \Phi_I\big]_{\text{surface}} \stackrel{!}{=} 0$, whereby the physical point-charge itself is removed from the configuration—and which therefore is not at all a solution to the physical configuration as originally given.

The other solution, $h' = +h$, is then the only non-trivial solution, and it places the image charge at exactly the same distance from the surface of the planar conductor as the physical point-charge, but on the opposite side, in the physically *inaccessible* region. This is indeed the solution we found through the observation #7 above.

It is useful to reconsider the above work and check whether the expansion in the region where $\rho < h, h'$ might give a different result. If so, it would imply that different portions of the boundary impose different conditions, which would not make sense physically. Using the results (A.5) with $a = \rho^2$ and $b = h^2$ and $b = (h')^2$, respectively:

$$0 \stackrel{!}{=} \frac{1}{4\pi\epsilon_0}\Big[\frac{q}{h}\Big(1 - \frac{1}{2}\frac{\rho^2}{h^2} + \frac{3}{8}\frac{\rho^4}{h^4} - \frac{5}{16}\frac{\rho^6}{h^6} + \cdots\Big) + \frac{q'}{h'}\Big(1 - \frac{1}{2}\frac{\rho^2}{(h')^2} + \frac{3}{8}\frac{\rho^4}{(h')^4} - \frac{5}{16}\frac{\rho^6}{(h')^6} + \cdots\Big)\Big], \tag{1.7}$$

$$= \frac{1}{4\pi\epsilon_0}\Big[\frac{qh'+q'h}{hh'} - \frac{\rho^2}{2}\frac{q(h')^3+q'h^3}{h^3(h')^3} + \frac{3\rho^4}{8}\frac{q(h')^5+q'h^5}{h^5(h')^5} - \frac{5\rho^6}{16}\frac{q(h')^7+q'h^7}{h^7(h')^7} + \cdots\Big]. \tag{1.8}$$

The vanishing of the ρ-independent term sets $q'h = -qh'$. Substituting this into the numerator of the second, ρ^2-term implies that

$$q(h')^3 + q'h^3 = q(h')^3 - qh'h^2 = qh'\big[(h')^2 - h^2\big]. \tag{1.9}$$

Since neither q nor h' vanish, this second ρ^2-term will vanish only if $h' = \pm h$. As before, the $h' = +h$ would place the image charge into the physically accessible location $\vec{r}' = (0,0,-h') = (0,0,h)$ and so is unphysical. In turn, the $h' = h$ solution then also implies that $q' = -q$.

Therefore, the solution does not depend on the form/domain of the expansion, (1.5) or (1.8). Indeed, one should always verify that the technical assumptions of a particular computation do not affect the physical content of the solution of that computation.

General Coordinates: One may wonder if the specific choice of the coordinate system (1.1) merely simplifies the computation, or somehow affects the result. On general grounds, we require that choosing a different coordinate system should *not* change the result (1.6) in a physically observable way, but this too is a property that is worth verifying.

So, consider the superposition expression $\Phi_{\text{tot.}}(\vec{r}) = \Phi_P(\vec{r}) + \Phi_I(\vec{r})$, written in a general coordinate system:

$$\Phi_{\text{tot.}}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\Big[\frac{q}{|\vec{r}-\vec{r}_0|} + \frac{q'}{|\vec{r}-\vec{r}'|}\Big]. \tag{1.10}$$

Setting $q' = -q$, the boundary condition becomes:

$$0 \overset{!}{=} \Phi_{\text{tot.}}(\vec{r}_s) = \frac{q}{4\pi\epsilon_0}\Big[\frac{1}{|\vec{r}_s-\vec{r}_0|} - \frac{1}{|\vec{r}_s-\vec{r}'|}\Big], \qquad \Rightarrow \quad |\vec{r}_s-\vec{r}_0| \overset{!}{=} |\vec{r}_s-\vec{r}'|. \tag{1.11}$$

That is, the image charge must be located so that the distance from it to every point on the surface of the planar conductor is equal to the distance from the physical point-charge to that same point in the surface. This can only happen if the image charge is placed at the point that is a mirror-reflection of the physical point-charge in the surface of the planar conductor:

1. on the opposite side of the surface (in the physically inaccessible region),
2. along the symmetry axis through the physical point-charge that is perpendicular to the surface, and
3. at the same distance from the surface.

This proves that the image charge configuration shown in the right-hand side images in Figures 1 and 2 is indeed the solution, regardless of the choice of the coordinates. In subsequent analyses, we then use symmetries maximally, as was done in the initial assessment for this configuration.

Surface Charges: Now use that the electrostatic potential produced by the image charge is in the physically accessible region equivalent (indistinguishable by any physical measurement) from the electrostatic potential produced by the surface charges. In particular, this produces the electrostatic field

$$\vec{E}(\vec{r}) = -\vec{\nabla}\,\Phi_{\text{tot.}}(\vec{r}) \tag{1.12}$$

which is simplest to evaluate in the "centered" cylindrical coordinate system as in the display (1.1):

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\Big[\frac{\rho\hat{e}_\rho + (z-h)\hat{e}_z}{[\rho^2+(z-h)^2]^{3/2}} - \frac{\rho\hat{e}_\rho + (z+h)\hat{e}_z}{[\rho^2+(z+h)^2]^{3/2}}\Big], \qquad z \geqslant 0, \tag{1.13}$$

Now apply Gauss' law, $\vec{\nabla}\cdot\vec{D} = \rho(\vec{r})$, to a cylindrical volume of which the top is immediately above the surface of the planar conductor, the bottom immediately below, and the sides form a cylinder of infinitesimal height that is perpendicular to the surface of the planar conductor, as in Eq. (1.22) [4, p. 31]:

$$\sigma(\rho,\phi) = \epsilon_0(\vec{E}_{\text{top}} - \vec{E}_{\text{bottom}})\cdot\hat{n}_s = \epsilon_0 \lim_{z\to 0^+} \hat{e}_z\cdot\vec{E}(\rho,\phi,z), \tag{1.14}$$

where the "surface of the planar conductor" is the actual, 2-dimensional surface separating the empty space above the conductor and the inside region of the conductor, however thin it may be. We identify $\hat{n}$ with the normal to the surface on the upper side, $\hat{n} = \hat{e}_z$ and note that $\vec{E}_{\text{bottom}}$ is the electrostatic field inside the planar conductor—which must be zero, since it would otherwise move charges inside the conductor and the configuration would not be static. Finally, the $z \to 0^+$ is computed *from above*, from the $z \geqslant$ region. This produces:

$$\sigma(\rho,\phi) = \epsilon_0 \lim_{z\to 0^+} \frac{q}{4\pi\epsilon_0}\Big[\frac{(z-h)}{[\rho^2+(z-h)^2]^{3/2}} - \frac{(z+h)}{[\rho^2+(z+h)^2]^{3/2}}\Big] = \frac{-q}{2\pi}\frac{h}{[\rho^2+h^2]^{3/2}}. \tag{1.15}$$

The total amount of surplus charge that has been attracted into the surface of the planar conductor equals

$$Q_{\sigma,\,\text{tot.}} = \int_0^\infty \rho \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\phi\; \sigma(\vec{r}) = -q \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \int_0^\infty \frac{h\,\rho \mathrm{d}\rho}{[\rho^2+h^2]^{3/2}},$$
$$\underset{s=h\rho}{=} \; -q \int_0^\infty \frac{s\mathrm{d}s}{[s^2+1]^{3/2}} \underset{u=s^2+1}{=} \; -q \int_1^\infty \frac{\frac{1}{2}\mathrm{d}u}{u^{3/2}} = \; -\frac{q}{2}\Big[\frac{-2}{\sqrt{u}}\Big]_1^\infty = q[0-1] = -q. \tag{1.16}$$

This implies that all of the electrostatic field emanating from the physical point-charge eventually sinks into the surface of the infinite plane conductor.

Further Computations: Once we have the electrostatic field—the potential (1.10) and the electric field (1.13) —we can compute other physical quantities too. For example, the force with which the physical point-charge attracts the planar conductor (holding the surface charges) is measurable:

$$\vec{F} = \int_0^\infty \rho \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\phi\; \sigma(\rho,\phi)\, \vec{E}_P(z{=}0) = \hat{\mathrm{e}}_z \int_0^\infty \rho \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\phi\; \sigma(\rho,\phi)\, E_{P,\,z}(z{=}0),$$
$$= \frac{\hat{\mathrm{e}}_z}{2\epsilon_0} \int_0^\infty \rho \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\phi\; \sigma(\rho,\phi)^2, \tag{1.17}$$

where $\vec{E}_P(\vec{r})$ is the part of the electrostatic field caused by the physical point-charge, equal to the first of the two contributions in (1.13), and the z-component of this produces $\frac{1}{2\epsilon_0}\sigma(\vec{r})$ according to the computation (1.15). Therefore:

$$\vec{F} = \frac{\hat{\mathrm{e}}_z}{2\epsilon_0} \int_0^\infty \rho \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\phi\; \frac{q^2}{4\pi^2} \frac{h^2}{[\rho^2+h^2]^3} = \frac{q^2 h^2 \hat{\mathrm{e}}_z}{4\pi\epsilon_0} \int_0^\infty \frac{\rho \mathrm{d}\rho}{[\rho^2+h^2]^3} = \frac{q^2 \hat{\mathrm{e}}_z}{4\pi\epsilon_0 h^2} \int_0^\infty \frac{s\mathrm{d}s}{[s^2+1]^3},$$
$$= \frac{1}{4\pi\epsilon_0} \frac{q^2}{(2h)^2} \hat{\mathrm{e}}_z. \tag{1.18}$$

By Newton's 3rd law, the same magnitude of force but in the opposite direction is exerted by the planar conductor on the physical point-charge, and that force equals precisely the force that the image charge $-q$ located at the distance $2h$ would exert. Therefore, the physically measurable effects of the physical planar conductor carrying the surface charges $\sigma(\vec{r})$ are indeed indistinguishable from those of the image charge $-q$ located at $(0,0,-h)$.

A Some Expansions

It will be useful to recall the power-expansion:

$$(a+b)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} a^k b^{\alpha-k}, \qquad \text{where} \quad \binom{\alpha}{k} := \frac{\alpha}{1}\cdot\frac{\alpha-1}{2}\cdot\frac{\alpha-2}{3}\cdots\frac{\alpha-k+1}{k}. \tag{A.1}$$

In particular,

$$\frac{1}{(a+b)} = \frac{1}{a}\sum_{k=0}^{\infty}(-1)^k\Big(\frac{b}{a}\Big)^k = \frac{1}{a}\Big[1-\frac{b}{a}+\frac{b^2}{a^2}-\frac{b^3}{a^3}+\cdots\Big], \qquad \text{if } |b|<|a|, \tag{A.2}$$

$$= \frac{1}{b}\sum_{k=0}^{\infty}(-1)^k\Big(\frac{a}{b}\Big)^k = \frac{1}{b}\Big[1-\frac{a}{b}+\frac{a^2}{b^2}-\frac{a^3}{b^3}+\cdots\Big], \qquad \text{if } |a|<|b|. \tag{A.3}$$

$$\frac{1}{\sqrt{a+b}} = \frac{1}{a}\sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}\Big(\frac{b}{a}\Big)^k = \frac{1}{\sqrt{a}}\Big[1-\frac{1}{2}\frac{b}{a}+\frac{3}{8}\frac{b^2}{a^2}-\frac{5}{16}\frac{b^3}{a^3}+\cdots\Big], \qquad \text{if } |b|<|a|, \tag{A.4}$$

$$= \frac{1}{b}\sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}\Big(\frac{a}{b}\Big)^k = \frac{1}{\sqrt{b}}\Big[1-\frac{1}{2}\frac{a}{b}+\frac{3}{8}\frac{a^2}{b^2}-\frac{5}{16}\frac{a^3}{b^3}+\cdots\Big], \qquad \text{if } |a|<|b|. \tag{A.5}$$

These results can also be reconstructed as the Taylor series, viewing the function $\frac{1}{(a+b)^\alpha}$ as the function $f(x)=x^{-\alpha}$, expanded near $x=a$, and where $|b|<|a|$ is a small shift:

$$\frac{1}{(a+b)^\alpha} = \sum_{k=0}^{\infty}\frac{b^k}{k!}\Big[\frac{\mathrm{d}^k(x^{-\alpha})}{\mathrm{d}x^k}\Big]_{x=a} = \sum_{k=0}^{\infty}\frac{b^k}{k!}\Big[\frac{\mathrm{d}^k(a^{-\alpha})}{\mathrm{d}a^k}\Big]. \tag{A.6}$$

The advantage of this latter method is that it readily admits a 3-dimensional generalization[9]:

$$f(\vec a+\vec b) = \underbrace{\sum_{k=0}^{\infty}\frac{1}{k!}\Big[\underbrace{\vec b\cdot\Big(\vec\nabla_a \cdots \vec b\cdot\big(\vec\nabla_a}_{k} f(\vec a)\big)\Big)\Big]}_{\text{when } |\vec b|<|\vec a|} = \underbrace{\sum_{k=0}^{\infty}\frac{1}{k!}\Big[\underbrace{\vec a\cdot\Big(\vec\nabla_b \cdots \vec a\cdot\big(\vec\nabla_b}_{k} f(\vec b)\big)\Big)\Big]}_{\text{when } |\vec a|<|\vec b|}. \tag{A.7}$$

Here, the del-operators $\vec\nabla_a$ are derivatives with respect to the components of the vector $\vec a$, while the del-operators $\vec\nabla_b$ are derivatives with respect to the components of the vector $\vec b$.

A simplifying but oft-needed special case occurs when $f(\vec r)=f(r)$ with $r:=|\vec r|$. Then,

$$\vec\nabla f(r) = \frac{\vec r}{r}\frac{\partial f}{\partial r}, \qquad \vec a\cdot\vec\nabla f(r) = \frac{\vec a\cdot\vec r}{r}\frac{\partial f}{\partial r}, \tag{A.8}$$

$$\begin{aligned}
\vec a\cdot\vec\nabla\Big(\vec a\cdot\vec\nabla f(r)\Big) &= \vec a\cdot\vec\nabla\Big(\frac{\vec a\cdot\vec r}{r}\frac{\partial f}{\partial r}\Big) = (\vec a\cdot\vec r)\,\vec a\cdot\Big[\vec\nabla\Big(\frac{1}{r}\frac{\partial f}{\partial r}\Big)\Big] + \big(\vec a\cdot\vec\nabla(\vec a\cdot\vec r)\big)\frac{1}{r}\frac{\partial f}{\partial r},\\
&= (\vec a\cdot\vec r)\,\vec a\cdot\Big[\frac{\vec r}{r}\frac{\partial}{\partial r}\Big(\frac{1}{r}\frac{\partial f}{\partial r}\Big)\Big] + \Big(\Big[a_x\frac{\partial}{\partial x}+a_y\frac{\partial}{\partial y}+a_z\frac{\partial}{\partial z}\Big][a_x x+a_y y+a_z z]\Big)\frac{1}{r}\frac{\partial f}{\partial r},\\
&= \frac{(\vec a\cdot\vec r)^2}{r^2}\Big[\frac{\partial^2 f}{\partial r^2}-\frac{1}{r}\frac{\partial f}{\partial r}\Big] + \big((a_x)^2+(a_y)^2+(a_z)^2\big]\big)\frac{1}{r}\frac{\partial f}{\partial r},\\
&= \frac{(\vec a\cdot\vec r)^2}{r^2}\Big[\frac{\partial^2 f}{\partial r^2}-\frac{1}{r}\frac{\partial f}{\partial r}\Big] + \vec a^{\,2}\frac{1}{r}\frac{\partial f}{\partial r} = \frac{(\vec a\cdot\vec r)^2}{r^2}\frac{\partial^2 f}{\partial r^2} + \frac{\vec a^{\,2}\vec r^{\,2}-(\vec a\cdot\vec r)^2}{r^3}\frac{\partial f}{\partial r},
\end{aligned} \tag{A.9}$$

and so on for the higher order terms in (A.7).

[9]Curiously, this simple but rather useful generalization of the standard Taylor series is seldom found in the literature; see however the unnumbered equation on p. 162 of Ref. [7].

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