0 Delta Function

0.1 Warm-Up: Cartesian Coordinates

The basic idea is really simple: in one dimension, the δ-function is defined to reduce the integration (a continuous summation) to a single point:

$$\int_{-\infty}^{\infty} \mathrm{d}x\; \delta(x-x_*)\, f(x) = f(x_*), \qquad \text{for any well-defined function, } f(x). \tag{0.1}$$

Appearances to the contrary, x here need not be the familiar Cartesian coordinate — the notion applies to any variable, although of course may well be *that,* familiar, Cartesian variable. If indeed, x denotes *the* Cartesian and we consider the 3-dimensional (volume) integral, then $\delta(x-x_*)$ reduces the x-directional integration without doing anything to the (independent!) y- and z-directional integrals:

$$\int \mathrm{d}^3\vec{r}\; \delta(x-x_*)\, f(\vec{r}) = \iiint \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\; \delta(x-x_*)\, f(x,y,z) = \iint \mathrm{d}y\, \mathrm{d}z\; f(x_*,y,z), \tag{0.2}$$

as long as the (here unwritten) integration domain of x does contain x_*. The $\delta(x-x_*)$-function reduces the 3-dimensional volume integration to a surface integration, over the vertical[1] (y,z)-plane located at the $x=x_*$ position along the x-axis. By the same token,

$$\int \mathrm{d}^3\vec{r}\; \delta(z-z_*)\, f(\vec{r}) = \iiint \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\; \delta(z-z_*)\, f(x,y,z) = \iint \mathrm{d}x\, \mathrm{d}y\; f(x,y,z_*) \tag{0.3}$$

reduces the 3-dimensional volume integration to a surface integration — *as long as the (here unwritten) integration domain of z does contain z_*.* — to the (horizontal) (x,y)-plane located at the $z=z_*$ height (along the customarily oriented z-axis).

Picture the originally 3-dimensional, *volume,* domain of integration on the far left-hand side of Eqs. (0.2) and (0.3) being reduced ("collapsed," "squelched") to a 2-dimensional, *surface,* subdomain of integration. The location of that subdomain is specified by the vanishing of the argument of the δ-function, thereby effectively enforcing that *constraint*:

1. The $\delta(x-x_*)$-function in (0.2) enforces the constraint $x-x_* \stackrel{!}{=} 0$, i.e., $x \stackrel{!}{=} x_*$.
2. The $\delta(z-z_*)$-function in (0.3) enforces the constraint $z-z_* \stackrel{!}{=} 0$, i.e., $z \stackrel{!}{=} z_*$.

That single constraint enforcement is the dimension-reducing effect of a single, 1-dimensional, δ-function.

Check your understanding: *Visualize the effect of the δ-function in the 3-dimensional integration $\int \mathrm{d}^3\vec{r}\; \delta(x-y)\, f(\vec{r})$ as a resulting 2-dimensional,* ***surface****-integration. Which way does that resulting surface lie within the 3-dimensional volume of the total, 3-dimensional space?*

Employing two δ-functions then straightforwardly reduces a 3-dimensional, *volume*-integral to a line integral:

$$\int \mathrm{d}^3\vec{r}\; \delta(x-x_*)\delta(y-y_*)\, f(\vec{r}) = \int \mathrm{d}z\; f(x_*,y_*,z), \tag{0.4}$$

[1] *Customarily,* we choose the z-axis vertically and the (x,y)-plane horizontally, but you are of course free to change that choice.

again assuming that x_* is within the (here unwritten) domain of the x-integration and y_* is within the (here unwritten) domain of the y-integration. For $x_* = 0 = y_*$, the resulting line-integral happens along the z-axis; if $x_* = 1 = y_*$, it happens along a straight line that is parallel to the z-axis, extending vertically and passing through the point $(x, y) = (1, 1)$ in the (x, y)-plane.

Finally, employing three δ-functions then clearly reduces a 3-dimensional, *volume*-integral to a point-"integral," i.e., evaluation at a point:

$$\int \mathrm{d}^3\vec{r}\;\delta(x-x_*)\delta(y-y_*)\delta(z-z_*)\,f(\vec{r}) = f(x_*, y_*, z_*). \tag{0.5}$$

So, before you try computing anything in the homework (or any other computation), establish what sort of integration-domain reduction is supposed to happen.

Check your understanding: *Visualize the effect of the δ-function in the 3-dimensional integration $\int \mathrm{d}^3\vec{r}\;\delta(x^2{+}y^2{-}R^2)\,f(\vec{r})$, with $0{<}R$ a constant. It must reduce to a 2-dimensional,* ***surface****-integration (with the dimensions of a surface-area). Which way does that resulting surface lie within the 3-dimensional volume of the total, 3-dimensional space?*

Separately, notice that the "physical" units of $\delta(F)$ must be the reciprocal of those of F — whatever F is — since $\int \mathrm{d}F\;\delta(F) = 1$, which follows from (0.1), on substituting $x \mapsto F$ and $f(\cdots) \mapsto 1$. Using "$[\cdots]$" as "units of $\cdots$," $\int \mathrm{d}F\;\delta(F) = 1$ implies $[F] = [\mathrm{d}F] = [\delta(F)]^{-1}$.

Check your understanding: *In the 3-dimensional integration $\int \mathrm{d}^3\vec{r}\;\delta(x^2{+}y^2{-}R^2)\,f(\vec{r})$, with $0{<}R$ a constant, use the usual Cartesian coordinates (x, y, z), each with units of length. If $f(\vec{r})$ and R both have physical dimensions of length, what must the units of this integral be?*

BTW, notice that if you change coordinates to cylindrical, $\delta(x^2{+}y^2{-}R^2) = \delta(\varrho^2{-}R^2)$, the argument, $\varrho^2{-}R^2$ of the δ-function has two zeros, i.e., the constraint equation $\varrho^2 \stackrel{!}{=} R^2$ has (formally) two solutions. However, only one of those can ever be within the geometrically meaningful domain of integration, with $\varrho \geqslant 0$.

0.2 Curvilinear Coordinates

Besides (ever so carefully) transforming the triple δ-function from (0.5) into cylindrical or spherical coordinates, there is a "quick trick," that simply uses the straightforward generalization of the defining property (0.1):

$$\int \mathrm{d}\vec{r}\;\delta^{(3)}(\vec{r}{-}\vec{r}_*)\,f(\vec{r}) = f(\vec{r}_*). \tag{0.6}$$

Notice that the 3-dimensional (triple) $\delta^{(3)}$-function reduces the 3-dimensional integral to a 0-dimensional, point-evaluation! (*This* is why it's *not* the $\delta^{(3)}$-function that's relevant to the homework problem 1.3.)

Writing out (0.6) in Cartesian coordinates recovers (0.5).

What *could* (0.6) then be in cylindrical coordinates?
We do know that $\mathrm{d}^3\vec{r} = \varrho\,\mathrm{d}\varrho\,\mathrm{d}\varphi\,\mathrm{d}\zeta$, and so (0.6) *must* have the form[2]:

$$\underbrace{\int_0^\infty \varrho\,\mathrm{d}\varrho \int_0^{2\pi} \mathrm{d}\varphi \int_{-\infty}^\infty \mathrm{d}z}_{\int \mathrm{d}^3\vec{r}}\ \underbrace{\Big(\boldsymbol{X}(\varrho,\varphi,z)\,\delta(\varrho{-}\varrho_*)\delta(\varphi{-}\varphi_*)\delta(z{-}z_*)\Big)}_{\delta^{(3)}(\vec{r}-\vec{r}_*)}\ f(\varrho,\varphi,z) = \tag{0.7}$$

$$\int_0^\infty \varrho\,\mathrm{d}\varrho\ \delta(\varrho{-}\varrho_*) \int_0^{2\pi} \mathrm{d}\varphi\ \delta(\varphi{-}\varphi_*) \int_{-\infty}^\infty \mathrm{d}z\ \delta(z{-}z_*)\ \boldsymbol{X}(\varrho,\varphi,z)\,f(\varrho,\varphi,z) \stackrel{!}{=} f(\varrho_*,\varphi_*,z_*), \tag{0.8}$$

where I "distributed" the three δ-functions to accompany the three (independent!) integrations, each of which will be reduced ("collapsed," "squelched"), and where I have a so-far-unknown factor, $\boldsymbol{X}(\varrho,\varphi,z)$, that I mentioned in class can be computed as the reciprocal of the Jacobian of the coordinate transformation from (0.5). The "trick" here is that we do not need to compute any Jacobians, since we know that:

$$\int_{-\infty}^\infty \mathrm{d}z\ \delta(z{-}z_*)\ \boldsymbol{X}(\varrho,\varphi,z)\,f(\varrho,\varphi,z) = \boldsymbol{X}(\varrho,\varphi,z_*)\,f(\varrho,\varphi,z_*), \tag{0.9}$$

for any value $z_* \in (-\infty,\infty)$ and since $[\delta(z{-}z_*)] = [\mathrm{d}z]^{-1}$, whereupon furthermore

$$\int_0^{2\pi} \mathrm{d}\varphi\ \delta(\varphi{-}\varphi_*)\ \boldsymbol{X}(\varrho,\varphi,z_*)\,f(\varrho,\varphi,z_*) = \boldsymbol{X}(\varrho,\varphi_*,z_*)\,f(\varrho,\varphi_*,z_*), \tag{0.10}$$

for any value $\varphi_* \in [0,2\pi]$ and since $[\delta(\varphi{-}\varphi_*)] = [\mathrm{d}\varphi]^{-1} = 1$; φ is dimensionless. This reduces (0.8) to

$$\int \mathrm{d}^3\vec{r}\ \delta^{(3)}(\vec{r}{-}\vec{r}_*)\,f(\vec{r}) = \int_0^\infty \varrho\,\mathrm{d}\varrho\ \delta(\varrho{-}\varrho_*)\ \boldsymbol{X}(\varrho,\varphi_*,z_*)\,f(\varrho,\varphi_*,z_*) \tag{0.11}$$

For (0.8) to hold and since $[\delta(\varrho{-}\varrho_*)] = [\mathrm{d}\varrho]^{-1}$, it must be that $[\boldsymbol{X}] = [\varrho]^{-1}$. Using again the (original, 1-dimensional) defining property (0.1),

$$\int \mathrm{d}^3\vec{r}\ \delta^{(3)}(\vec{r}{-}\vec{r}_*)\,f(\vec{r}) = \int_0^\infty \mathrm{d}\varrho\ \delta(\varrho{-}\varrho_*)\ \Big(\varrho\,\boldsymbol{X}(\varrho,\varphi_*,z_*)\,f(\varrho,\varphi_*,z_*)\Big), \tag{0.12}$$

$$= \varrho_*\,\boldsymbol{X}(\varrho_*,\varphi_*,z_*)\,f(\varrho_*,\varphi_*,z_*) \stackrel{!}{=} f(\varrho_*,\varphi_*,z_*), \tag{0.13}$$

so that $\boldsymbol{X}(\varrho_*,\varphi_*,z_*) = \frac{1}{\varrho_*}$, and therefore $\boldsymbol{X}(\varrho,\varphi,z) = \frac{1}{\varrho}$. This implies that the 3-dimensional δ-function in cylindrical coordinates is best thought of as

$$\delta^{(3)}(\vec{r}{-}\vec{r}_*) = \delta(\varrho{-}\varrho_*)\ \tfrac{1}{\varrho}\delta(\varphi{-}\varphi_*)\ \delta(z{-}z_*) \tag{0.14}$$

where we distributed the $\frac{1}{\varrho}$ factor so that each of the three factors has the dimensions of 1/length — just as do the three Cartesian δ-functions in (0.5); for a rigorous derivation, see § 3.10 of Arfken-Weber-Harris' text. In turn, the 3-dimensional, *volume*-integration over a 3-dimensional (triple) δ-function in cylindrical coordinates simplifies to:

$$\int \mathrm{d}^3\vec{r}\ \delta^{(3)}(\vec{r})\,f(\vec{r}) = \int_0^\infty \mathrm{d}\varrho\ \delta(\varrho{-}\varrho_*) \int_0^{2\pi} \mathrm{d}\varphi\ \delta(\varphi{-}\varphi_*) \int_{-\infty}^\infty \mathrm{d}z\ \delta(z{-}z_*)\ f(\varrho,\varphi,z). \tag{0.15}$$

In hindsight, the result (0.15) may look "obvious"[3]: the 3-dimensional (triple) δ-function must collapse all three integrations to point-evaluations — and must also cancel the (Jacobian) ϱ-factor in the cylindrical volume-element, $\mathrm{d}^3\vec{r} = \varrho\,\mathrm{d}\varrho\,\mathrm{d}\varphi\,\mathrm{d}z$.

---

[2] I will now explicitly integrate over all available space.

[3] *Everything* is obvious in hindsight.

Check your understanding: *Visualize the effect of the δ-function in the 3-dimensional integration in cylindrical coordinates, $\int \mathrm{d}^3\vec{r}\, \frac{1}{\varrho}\delta(\varphi{-}\pi)\, f(\vec{r})$, as a resulting 2-dimensional, **surface**-integration. Which way does that resulting surface lie within the 3-dimensional volume of the total, 3-dimensional space? For $f \to 1$, this must recover a surface area formula.*

Curvilinear coordinates are known to have "funny" locations. For example, the elementary, infinitesimal volume element $\varrho\,\mathrm{d}\varrho\,\mathrm{d}\varphi\,\mathrm{d}z$ literally vanishes along the z-axis, where $\varrho{=}0$. This may seem a peculiar "oddity" of not much relevance, as one might correlate this with the fact that the (3-dimensional) volume of the z-axis is indeed zero. However, since the same volume element $\varrho\,\mathrm{d}\varrho\,\mathrm{d}\varphi\,\mathrm{d}z$ does *not* vanish identically along any other vertical line, its vanishing along the z-axis cannot possibly have anything to do with the vanishing of the (3-dimensional) volume of any line. This "oddity" is referred to as a "coordinate singularity": among other things, along the z-axis, the value of the φ-angle is ill-defined: it may be computed from the Cartesian coordinates as

$$\varphi = \tan^{-1}(y/x) + \vartheta(-x)\pi + \vartheta(z)\vartheta(-y)2\pi, \qquad \vartheta(a) := \begin{cases} 1, & \text{for } a \geqslant 0; \\ 0, & \text{for } a < 0; \end{cases} \tag{0.16}$$

which makes it obvious that φ is ill-defined along the z-axis, where $x{=}0{=}y$. (ϑ is the step function, and those two ϑ-terms insure that φ takes on the successive values $\varphi \in [0, 2\pi]$ as one traverses the four quadrants of the (x, y)-plane in the usual, counterclockwise fashion.)

With (0.14) and (0.15) understood, the standard 3-dimensional integration is easily restricted to the "coordinate surfaces":

1. Fixed radius (cylinder): $\int \mathrm{d}^3\vec{r}\, \delta(\varrho{-}R)\, f(\vec{r}) = R \int \mathrm{d}\varphi \int \mathrm{d}z\ f(R, \varphi, z)$.
2. Fixed angle (semi-plane[4]): $\int \mathrm{d}^3\vec{r}\, \frac{1}{\varrho}\delta(\varphi{-}\varphi_*)\, f(\vec{r}) = \int \mathrm{d}\varrho \int \mathrm{d}z\ f(\varrho, \varphi_*, z)$.
3. Fixed height (plane): $\int \mathrm{d}^3\vec{r}\, \delta(z{-}h)\, f(\vec{r}) = \int \varrho\,\mathrm{d}\varrho \int \mathrm{d}\varphi\ f(\varrho, \varphi, h)$.

Notice that all 2-dimensional integrals have the physical units of a surface area. Reducing the default, "all space" domains $\varrho \in [0, \infty)$, $\varphi \in [0, 2\pi]$ and $z \in (-\infty, \infty)$ to appropriate sub-regions in the above integrations then reduces the integration to finite geometrical objects such as

1. upright cylinder: $R \int_0^{2\pi} \mathrm{d}\varphi \int_0^h \mathrm{d}z\ f(R, \varphi, z)$,
2. circular disc: $\int_0^R \varrho\,\mathrm{d}\varrho \int_0^{2\pi} \mathrm{d}\varphi\ f(\varrho, \varphi, h)$,

and so on. Notice that the so-defined "coordinate surface"-integrals and corresponding factors of the 3-dimensional δ-function satisfy the intermediate, 2-dimensional version of the defining property, such as:

$$\int \varrho\,\mathrm{d}\varrho \int \mathrm{d}\varphi\ \delta(\varrho{-}\varrho_*)\ \tfrac{1}{\varrho}\delta(\varphi{-}\varphi_*)\, f(\varrho, \varphi, h) = f(\varrho_*, \varphi_*, h). \tag{0.17}$$

Combining two δ-functions results in "coordinate line"-integrals such as, e.g.,

$$\int \mathrm{d}^3\vec{r}\, \delta(\varrho{-}R)\, \tfrac{1}{\varrho}\delta(\varphi{-}\varphi_*)\, f(\vec{r}) = \int_0^L \mathrm{d}z\ f(R, \varphi_*, z), \tag{0.18}$$

[4] This planar surface is bounded on one side by the z-axis and extends along φ_* to infinity.

for a vertical length-L line at (R,φ_*), whereas

$$\int \mathrm{d}^3\vec{r}\;\delta(\varrho{-}R)\delta(z{-}h)\,f(\vec{r}) = R\int_0^{2\pi}\mathrm{d}\varphi\;f(R,\varphi,h), \tag{0.19}$$

for a horizontal radius-R circle at height h, and so on.

Check your understanding: *Following the above reasoning and* (0.14)*, justify that the 3-dimensional (triple)* δ*-function in spherical coordinates must be*

$$\delta^{(3)}(\vec{r}{-}\vec{r}_*) = \rho(r{-}r_*)\;\tfrac{1}{r}\delta(\theta{-}\theta_*)\;\tfrac{1}{r\sin(\theta)}\delta(\varphi{-}\varphi_*). \tag{0.20}$$

Again, the factors all have units of 1/length, and the $1/\sin(\theta)$*-factor is included with the* $\delta(\varphi)$*-function since the* φ*-angle is the same as in the cylindrical coordinates, and* $\varrho = r\sin(\theta)$*. For a rigorous derivation, see § 3.10 of Arfken-Weber-Harris' text.*

From this, again following the above reasoning, deduce the form of the surface integrals over:
(1) *a radius-*R*, origin-centered spherical shell,* ***(2)*** *an upward-convex cone of opening angle* 2α *and apex-to-rim radius* R*.*
Hint: For each of these surface-integrals, a 2-dimensional analogues of (0.17) *must hold.*

By setting the integrated function to 1, re-derive the known formulae for the surface area of ***(1)*** *a height-*h*, radius-*R*, capped at both ends cylindrical shell,* ***(2)*** *a radius-*R *spherical shell, as well as the volumes contained by these.*

Oh, well; here's that conical integral:

$$\int_0^R r^2\mathrm{d}r\int_0^\pi \sin(\theta)\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi\;\tfrac{1}{r}\delta(\theta{-}\alpha)\;f(r,\theta,\varphi) = \sin(\alpha)\int_0^R r\mathrm{d}r\int_0^{2\pi}\mathrm{d}\varphi\;f(r,\alpha,\varphi). \tag{0.21}$$

If $f(r,\alpha,\varphi)=\sigma$ is a constant, uniform surface charge density (units Coulomb/m^2), then the total charge is $Q=\sin(\alpha)(R^2/2)(2\pi)\sigma=\pi R^2\sin(\alpha)\sigma$, which for $\alpha=\pi/2$ (equatorial plane, i.e., flat disc) recovers the $Q=\pi R^2\sigma$ result, i.e., $\sigma=Q/(\pi R^2)$, as expected. If in turn the cone "closes," $\alpha\to 0$ (North Pole) or $\alpha\to\pi$ (South Pole), the surface area also limits to 0, and the surface charge density "blows up" as $\sigma\to 1/0$ — which we will see makes sense for the behavior of "spiky" charged objects.