Stringy de Sitter Brane-Worlds

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Dedicated to the memory of Pavao Senjanović

The possibility that our 3+1-dimensional world might be a cosmic defect (brane-world) within a higher-dimensional spacetime[1] has recently attracted much interest, owing to the proof [7] that gravity may be localized on such brane-worlds. Randall and Sundrum showed that such geometries may also solve the hierarchy problem [8]. However, it remained unclear whether these and other desirable properties can be achieved within the *same* model.

Herein, we describe a family of stringy toy model brane-worlds [1, 2, 3, 4, 5, 6], which generalize the concept of spacetime variable cosmic strings [9, 10] and exhibits *simultaneously*:

1. exponential hierarchy of Plank mass scales,
2. localized gravity on the brane-world,
3. an induced de Sitter metric on the brane-world,
4. a phenomenologically acceptable value for the cosmological constant,
5. a dynamical mechanism for either trapping the bulk-roaming degrees of freedom to the brane-world, or decoupling them from it,

and where the spacetime geometry is driven by the anisotropy of the axion-dilaton moduli field. Furthermore, the axion-dilaton background configuration possesses crucial stringy $SL(2, \mathbf{Z})$ monodromy, and many of the features are a direct and quantifiable consequence of supersymmetry breaking.

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[1] For a fairly complete bibliography on the subject, see Refs. [1, 2, 3, 4, 5, 6].

1 A Stringy Family of Toy Models

We begin with a higher-dimensional string [12, 13] or F-theory [11] compactified on a Calabi-Yau (complex) n-fold, some moduli (ϕ^α) of which are allowed to vary over (the 'transversal') part of the non-compact space. Following Refs. [9, 10], the effective action describing the coupling of the moduli to gravity of the observable spacetime is derived by dimensionally reducing the higher dimensional Einstein-Hilbert action. The relevant part of the low-energy effective D-dimensional action of the moduli, ϕ^α, of the Calabi-Yau n-fold coupled to gravity then reads:

$$S_0 + S^b_{\rm eff} = \frac{1}{2\kappa^2}\int {\rm d}^D x\sqrt{-g}(R - \mathcal{G}_{\alpha\bar\beta}g^{\mu\nu}\partial_\mu\phi^\alpha\partial_\nu\phi^{\bar\beta} + ...) + S^b_{\rm eff}\ . \qquad (1)$$

Here $\mu,\nu = 0,\cdots,D-1$, $2\kappa^2 = 16\pi G^D_N$, where G^D_N is the D-dimensional Newton constant, and $\mathcal{G}_{\alpha\bar\beta}$ is the metric on $\mathcal{M}_\phi$, the space of moduli ϕ^α. Higher derivative terms and all other fields in the theory are neglected. We also restrict the moduli to depend on the *'transversal'* coordinates, x_i, $i{=}D{-}2, D{-}1$, and so have a vanishing *'longitudinal'* gradient: $\partial_a\phi{=}0$, $a{=}0,\cdots,D{-}3$. $S^b_{\rm eff}$ is a purely $(D{-}1)$-dimensional effective action describing the (our?) brane-world implied by the explicit form of the solution described below. The moduli obey the equation of motion:

$$g^{ij}\Big(\nabla_i\nabla_j\phi^\alpha + \Gamma^\alpha_{\beta\gamma}(\phi,\bar\phi)\partial_i\phi^\beta\partial_j\phi^\gamma\Big) = 0\ , \qquad (2)$$

where $\Gamma^\alpha_{\beta\gamma}$ is the Christoffel connection on $\mathcal{M}_\phi$. Note that $S^b_{\rm eff}$ does not depend on the moduli. The Einstein equations are:

$$R_{\mu\nu} - \tfrac{1}{2}2g_{\mu\nu}R = T_{\mu\nu}(\phi,\bar\phi) + T^b_{\mu\nu}\ , \qquad (3)$$

$$T_{\mu\nu} = \mathcal{G}_{\alpha\bar\beta}\Big(\partial_\mu\phi^\alpha\partial_\nu\phi^{\bar\beta} - \tfrac{1}{2}2g_{\mu\nu}\,g^{\rho\sigma}\partial_\rho\phi^\alpha\partial_\sigma\phi^{\bar\beta}\Big)\ , \qquad (4)$$

where $T^b_{\mu\nu}$ is a delta-function source, as shown below.

1.1 Matter

For our family of toy models, we *choose*: $\phi^\alpha = \tau := {\rm a}{+}ie^{-\Phi}$, representing the axion-dilaton system of the $D{=}10$ Type IIB string theory, thought of as a T^2-compactification of F-Theory, so $\mathcal{G}_{\tau\bar\tau} = [\Im m(\tau)]^{-2}$ is the Teichmüller metric [11]. Now, we *assume* that $\tau = \tau(\theta)$, where $\theta = \arctan(\frac{x_{D-1}}{x_{D-2}})$ is the "polar" angle in the transversal (x_{D-2}, x_{D-1})-plane. With this, Eq. (2) becomes:

$$\tau'' + \frac{2}{\bar\tau - \tau}\ =\ 0\ , \qquad (5)$$

and is solved by:

$$\tau_I(\theta) = \mathrm{a}_0 + i\, g_s^{-1}\, e^{\omega(\theta-\theta_0)}\ , \qquad \mathrm{a}_0, g_s, \omega, \theta_0 = const\ , \tag{6}$$

$$\tau_{II}(\theta) = \mathrm{a}_0 \pm g_s^{-1}\, \frac{\sinh[\omega(\theta-\theta_0)] + i}{\cosh[\omega(\theta-\theta_0)]}\ , \tag{7}$$

which satisfies our *requirement* that its energy-momentum tensor be a constant[2], $\propto \omega^2$. Both solutions are discontinuous across the branch-cut, $(\theta-\theta_0) = \pm\pi$, but the constants $\mathrm{a}_0, g_s, \omega$ may be chosen so that $\tau(\theta_0+\pi) = M{\cdot}\tau(\theta_0-\pi)$, where $M \in SL(2,\mathbf{Z})$. That is, both τ_I and τ_{II} exhibit (different) non-trivial $SL(2,\mathbf{Z})$ *monodromy* [1, 2].

The absence (in the limit of exact supersymmetry) of a potential for τ, and their nontrivial $SL(2,\mathbf{A})$ monodromy enforces the conclusion:

The metric-moduli system (1), (6-7) can only stem from a string theory.

1.2 Minkowski Metric

With a phenomenologically interesting $K3$ compactification of the $D{=}10$ solution in the back of our minds (upon which the metric receives α' corrections), we however keep D unspecified for the sake of generality. The metric that interpolates between the two solutions of Ref. [1, 2], with $z = \log(r/\ell)$, is:

$$\mathrm{d}s^2 = A(z)\eta_{ab}\mathrm{d}x^a\mathrm{d}x^b + \ell^2 B(z)(\mathrm{d}z^2 + \mathrm{d}\theta^2)\ , \tag{8}$$

$$A(z) = Z^{\frac{2}{D-2}}\ , \qquad Z(z) := 1 + a_0|z|\ , \tag{9}$$

$$B(z) = Z^{-\frac{D-3}{D-2}}\, e^{\frac{\xi}{a_0}(\beta - Z^2)}\ , \tag{10}$$

Here ξ, a_0 and β are free parameters, $\ell \sim O(M_D^{-1})$ sets the transversal length scale, and η_{ab} is the Minkowski metric along the $(D{-}2)$-dimensional brane-world. The dependence on $|z|$ (in place of just z in Refs. [1, 2]) induces the δ-function terms in Eq. (3) (with $\varrho := a_0 e^{-\frac{\xi}{a_0}(\beta-1)}[\frac{D-3}{D-2} - 2\frac{\xi}{a_0}]$)

$$-\eta_{ab}\,\ell^{-2}\Big[a_0\xi\,\mathrm{sign}^2(z)\,Z^{\frac{D-1}{D-2}}\,e^{-\frac{\xi}{a_0}(\beta-Z^2)} - \varrho\,\delta(z)\Big] = T_{ab} + T^b_{ab}\ , \tag{11}$$

$$-a_0\xi\,\mathrm{sign}^2(z) = T_{zz} + T^b_{zz}\ , \tag{12}$$

$$+a_0\xi\,\mathrm{sign}^2(z) + 2a_0\,\delta(z) = T_{\theta\theta} + T^b_{\theta\theta}\ . \tag{13}$$

Hereafter, we refer to the brane at $z = 0$ as the *brane-world*: there, $\mathrm{sign}^2(z) = 0$ and so $T_{\mu\nu} = 0$ also. On the other hand, the δ-function terms in the left-hand side of the Einstein equations (3) are now non-zero and read:

$$T^b_{\mu\nu}\ =\ \ell^{-2}\,\mathrm{diag}\Big[-\varrho, \varrho, \ldots, \varrho, 0, 2a_0\Big]\,\delta(z)\ . \tag{14}$$

Since τ depends on ω (which Eqs. (11–13) fix to $\omega^2 \equiv 8a_0\xi \geq 0$), we see that $\varrho \geq 0$ and $\mathrm{sign}(T^b_{00}) = -\mathrm{sign}(a_0)$ for $|\xi| \leq |\xi_c| := \frac{1}{2}\frac{D-3}{D-2}|a_0|$. In particular,

[2] Requiring that $\tau = \tau(\theta)$ and that its energy-momentum tensor be constant permits solving for the metric as independent of θ by means of separation of variables.

$T^b_{00} \geq 0$ in the $a_0 \leq 0$ case, when z is restricted between the naked (null) singularities at $z = \pm 1/a_0$. When $\xi = 0$ this form of the stress tensor is similar to that of spatial domain walls [14, 15], in which the surface energy density, σ, is equal to the surface tension, $-p$, where p is the pressure along the domain wall. In our case, however $|T^b_{\theta\theta}| > T^b_{00}$. From this it follows that the weak energy condition holds except for $T^b_{\theta\theta}$, *i.e.*, $T^b_{\mu\nu}\zeta^\mu\zeta^\nu < 0$ only for the null vector $\zeta^\mu = (1, 0, \cdots, 0, \sqrt{A/B})$ representing a vortex in the transversal (z, θ)-plane. (For a related discussion of this feature of co-dimension two solutions consult for example [16].) Still, we *assume* that it is possible to associate an effective action for the source at $z = 0$,

$$S^b_{\text{eff}} = \int \mathrm{d}^{D-2}x \, \mathrm{d}z \, \mathrm{d}\theta \, \sqrt{-g} \, \delta(z) \, \lambda \, \mathcal{L}^b \ , \tag{15}$$

depending on all matter localized[3] to this (our?) brane-world. Equating the $T^b_{\mu\nu}$ calculated from (15) with the δ-function contribution of the Einstein equations from Eq. (11–13), we obtain that

$$\lambda \sim -a_0 \ell^{-2} e^{-\frac{\xi}{a_0}(\beta-1)} \qquad |\xi| \ll |\xi_c| \ . \tag{16}$$

Note that the vacuum energy which couples to gravity is $\lambda \mathcal{L}^b = -\lambda$. Analogous results hold also in the $a_0 > 0$ and $\xi > \xi_c$ case. However, now $T^b_{\mu\nu}\zeta^\mu\zeta^\nu > 0$ for *all* null vectors. We thus have two subfamilies of solutions:

1. $a_0, (|\xi| - |\xi_c|) < 0$, where $T^b_{\mu\nu}\zeta^\mu\zeta^\nu < 0$ only for $\zeta^\mu = (1, 0, \cdots, 0, \sqrt{A/B})$, the brane-world is encircled by naked singularities at $z = \pm 1/a_0$;
2. $a_0, (|\xi| - |\xi_c|) > 0$, where $T^b_{\mu\nu}\zeta^\mu\zeta^\nu > 0$ for *all* null vectors, and the transverse space is infinite, with temporal singularities at $z = \pm\infty$.

Incidentally, replacing η_{ab} with *any Ricci-flat metric* (*e.g.*, the Schwarzschild geometry), leaves the above solutions unchanged.

1.3 de Sitter Metric

Now modify Eq. (8) into:

$$\mathrm{d}s^2 = \tilde{A}^2(z) \, \tilde{g}_{ab} \, \mathrm{d}x^a \mathrm{d}x^b + \ell^2 \tilde{B}^2(z) \, (\mathrm{d}z^2 + \mathrm{d}\theta^2) \ , \tag{17}$$

$$[\tilde{g}_{ab}] = \mathrm{diag}[-1, e^{2\sqrt{\Lambda}x^0}, \cdots, e^{2\sqrt{\Lambda}x^0}] \ , \tag{18}$$

where Λ is the cosmological constant for the brane-world spacetime. (Note that there is no cosmological constant in D-dimensional spacetime of the original string or F-theory!)

The closed-form solutions (9–10) no longer apply. Instead, the purely longitudinal part of Eqs. (3) reduces to a single equation, giving:

[3] Localization of matter is a generic feature in superstring theories [17].

$$\tilde{B}^2 = \ell^{-2} \Lambda^{-1} \frac{h'' h^{-\frac{D-4}{D-2}}}{(D-2)} , \qquad h(z) := \tilde{A}(z)^{D-2} , \tag{19}$$

which determines $\tilde{B}(z)$ in terms of $h(z)$ and so $\tilde{A}(z)$. Upon this substitution, the remaining components of Eqs. (3) produce the following single equation[4]:

$$\frac{1}{2(D-2)} \frac{h'^2}{h^2} - \frac{h''}{2h} + \frac{h' h'''}{2hh''} + \frac{1}{8}\omega^2 \; = \; 0 \; . \tag{20}$$

This implies that $\Lambda>0$, and that the Ansatz (17–18) does not permit a double Wick rotation into an anti-de Sitter spacetime, and conversely that our solution cannot be obtained from any anti-de Sitter solution of string theory. To see this, note that Eq. (20) determines $h(z)$, and hence $\tilde{A}(z)$, to be independent of Λ. But then, $\Lambda \to -\Lambda$ in Eq. (19) would imply $\tilde{B}(z)^2 < 0$, making the entire plane transverse to the cosmic brane also time-like.

Furthermore, with $h(z) = (1 - z/z_0)^{D-2}$, and so with

$$\tilde{A}_0(z) = \tilde{Z}(z) := (1 - z/z_0) \, , \quad \text{and} \quad \tilde{B}_0(z) := \frac{1}{\ell z_0 \sqrt{\Lambda}} \; , \tag{21}$$

the metric (17) satisfies the Einstein equations (3) for $\omega^2 = 0$, *i.e.*, when $\tau = const.$ This solution describes the familiar Rindler space [18].

For $\omega \neq 0$ ($\tau \neq const.$), Eq. (20) has a perturbative solution[5] by expanding around the horizon, $\tilde{Z}(z) = 0$:

$$\tilde{A}(z) = \tilde{Z}(z) \Big(1 - \frac{\omega^2 z_0^2 (D-3)}{24(D-1)(D-2)} \tilde{Z}(z)^2 + O(\omega^4) \Big) \; , \tag{22}$$

$$\tilde{B}(z) = \frac{1}{\ell z_0 \sqrt{\Lambda}} \Big(1 - \frac{\omega^2 z_0^2}{8(D-1)} \tilde{Z}(z)^2 + O(\omega^4) \Big) \; . \tag{23}$$

Notice that, depending on $\tilde{A}(z)^2$ and $\tilde{B}(z)^2$, the metric (17) is well-defined for all values of z, with merely a *horizon* [19, 20] at $z = z_0$. It is easy to check that for our solution (22–23) both the Ricci scalar and tensor vanish at $z = z_0$, as does the whole Riemann tensor. In fact, these tensors as well as the $R_{\mu\nu} R^{\mu\nu}$ and $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ curvature scalars all remain bounded for all finite z. So, close to the horizon spacetime is asymptotically flat in agreement with the behavior of Rindler space, see Eqs. (21)–(23) [18]. However, the horizon does provide an effective cut-off of spacetime and, as usual in de Sitter space, we will only consider the degrees of freedom inside this horizon.

In contrast, when $\Lambda = 0$, the solution (9–10) with $a_0<0$ exhibits a naked singularity, at $z = \pm a_0^{-1}$ ($Z = 0$), for the global cosmic brane and the region

[4] It is straightforward to show that R_{zz} and $R_{\theta\theta}$ can be written as certain linear combinations of the left-hand side of the differential equation (20) and its derivatives.

[5] This solution is of the same form as that discussed by Gregory [19, 20] for the $U(1)$ vortex solution.

$|z| > |a_0^{-1}|$ ($Z < 0$) is unphysical: the metric becomes complex. In comparing (9) with the de Sitter solution (22), *the singularity is effectively removed* by introducing a non-zero longitudinal cosmological constant. Note that in solving (20), $h'' \neq 0$ was assumed. But $\Lambda \to 0$ implies that $h'' \to 0$, which gives rise to the solution of $A(z)$ in Eq. (9), with Eq. (19) no longer valid.

While the naked singularity of the Minkowski solution (9–10) has been removed by the non-zero Λ, away from the horizon this Minkowski solution is still a good approximation to Eq. (23). To compare, we first obtain a power series solution of Eqs. (23), expanding around the core, at $z = 0$. From this we determine the lowest order terms[6] in $h(z) = \sum_{n=0} h_n z^n$. Finally, we expand $\tilde{A}(z)$ and $\tilde{B}(z)$, expressed as functions of $h(z)$ and $h''(z)$ to lowest order in z,

$$\tilde{A}(z) = (1{+}z\frac{h_1}{(D{-}2)}) \;, \quad \tilde{B}(z) = \sqrt{\frac{2h_2}{(D{-}2)\Lambda\ell^2}}\Big[1{+}z\Big(\frac{3h_3}{2h_2}{-}\frac{h_1(D{-}4)}{2(D{-}2)}\Big)\Big] \;. \tag{24}$$

Here, the coefficients h_i for $i > 2$ are determined in terms of h_0, h_1, h_2 by Eq. (20), the overall rescaling of $\tilde{A}(z)$ and $\tilde{B}(z)$ is absorbed in a rescaling of x^a and ℓ, respectively, and the numerical values of h_1, h_2 are determined by comparison with the expansions (23). Comparing now Eq. (24) with Eq. (9–10), expanded to first order in z, leads to

$$a_0 \approx 0.9\,\frac{(D{-}2)}{\rho_0} \;, \quad \xi \approx \frac{1}{0.9}\frac{\omega^2\rho_0}{8(D-2)} \;, \quad \text{and} \quad \frac{\omega^2}{2(D-2)\Lambda\ell^2} = 1 \;. \tag{25}$$

The last of the identifications (25) implies:

$$\Lambda = \frac{\omega^2}{2(D-2)\ell^2} \;, \tag{26}$$

thus *expressing the cosmological constant in terms in the brane-world of the transversal anisotropy of the axion-dilaton system*! This gives a very non-trivial relation between the stringy moduli, and hence string theory itself, and a *positive* cosmological constant Λ. Since the dilaton is $\Phi = -\omega\theta$ it also follows from Eq. (26) that we have a strongly coupled theory[7].

Note also that $\Lambda \sim \omega^2/\ell^2$ implies that supersymmetry breaking and a non-zero cosmological constant are related: In our family of toy models, supersymmetry is explicitly broken by $\omega^2 \neq 0$. But since $\Lambda \sim \omega^2/\ell^2$, supersymmetry breaking by $\omega^2 \neq 0$ also induces a positive cosmological constant,

[6] This requires an initial guess for the value of $\omega^2\rho_0^2$ and that the higher order corrections in the expansion of $\tilde{A}(z)$ in terms of $\tilde{Z}(z)$ fall off fast enough. Indeed, we have computed the expansion of $\tilde{A}(z)$ to $O(\tilde{Z}^{12}(z))$, and determined $\omega^2\rho_0^2$ and the corresponding numerical values of the coefficients h_i recursively.

[7] Recall: with $\tau = a_0 + i g_s^{-1}\exp(\omega\theta)$, the $SL(2,\mathbf{Z})$ monodromy sets $g_s^D \sim O(1)$ in D dimensions. However, in the $D{-}2$-dimensional brane-world, $g_s^{D-2} = g_s^D\sqrt{\alpha'/V_\perp}$, and since the volume of the transverse space, $V_\perp$, is large (27), $g_s^{D-2} \ll 1$.

which then can vanish only in the decompactifying limit, $\ell \to \infty$. In the limit $\omega^2 = 0$ we recover supersymmetry and thus have a possible (supersymmetric) F-theory [11] background.

The cosmological constant on the brane-world is thus induced by the supersymmetry breaking caused by the anisotropy of the axion-dilaton system.

2 Localization of Gravity and Planck Mass

Unlike in the original Randall-Sundrum models [7, 8] (and, to the best of my knowledge, also any other brane-world model), for a suitable choice of parameters, the above family of toy models exhibits *both* an exponentially large hierarchy *and* localized gravity [4].

2.1 Exponential hierarchy

The large hierarchy between the $(D-2)$- and D-dimensional Planck scales is the same as in Refs. [1, 2]:

$$M_{D-2}^{D-4} = M_D^{D-2} \int_{M_\perp} \mathrm{d}v\, \mathrm{d}\theta\ \psi_0^2(v) = M_D^{D-2} \frac{2\pi\ell^2}{|a_0|} e^{\frac{\beta\xi}{a_0}} \left(\frac{a_0}{\xi}\right)^{\frac{D-3}{2(D-2)}} I^D_{a_0,\xi}\ , \quad (27)$$

$$I^D_{a_0,\xi} = \begin{cases} \left[\Gamma\left(\frac{D-3}{2(D-2)}\right) - \gamma\left(\frac{D-3}{2(D-2)}; \frac{\xi}{a_0}\right)\right] & \text{for } a_0 > 0, \\ \gamma\left(\frac{D-3}{2(D-2)}; \frac{\xi}{a_0}\right) & \text{for } a_0 < 0. \end{cases} \quad (28)$$

where $M_\perp$ denotes the hyperbolic transverse space [2]. Note that the large hierarchy is controlled by the product of β and the ratio $\frac{\xi}{a_0} > 0$, where the positivity of the latter is due to the presence of the non-trivial stringy moduli. It is therefore possible to choose $\frac{\beta\xi}{a_0}$, so as to have a large hierarchy between M_D and M_{D-2}.

Following the discussion of Randall and Sundrum [8] we compute the coupling of gravity to the fields on the brane. Writing, $\bar{g}_{\mu\nu} := g_{\mu\nu}|_{z=0}$ for the metric on the brane-world, there is a non-trivial contribution from $\sqrt{-g}$, *i.e.*, $\sqrt{-g}|_{z=0} = \sqrt{-\bar{g}}\,\ell^2 e^{(\beta-1)\xi/a_0}$. Hence, (15) becomes

$$S^b_{\text{eff}} \sim -a_0 \int \mathrm{d}^{D-2}x\mathrm{d}\theta\ \sqrt{-\bar{g}}\,\mathcal{L}^b\ , \quad (29)$$

where we have taken into account the tension for the brane-world according to Eq. (16). Thus, unlike in Ref. [8], here the fields, masses, couplings and vev's in $\mathcal{L}^b$ retain their fundamental, D-dimensional value, $O(M_D)$. Also, using Eq. (8), the kinetic terms of a typical field, Ψ, expand

$$|\partial_\mu \Psi|^2 = |\partial_\| \Psi|^2 + \ell^{-2} e^{-\frac{\xi}{a_0}(\beta-1)} |\partial_\perp \Psi|\ , \quad (30)$$

so that the transverse excitations of Ψ are exponentially suppressed.

2.2 Localization of gravity

To understand the localization of gravity, we look at small gravitational fluctuations $\delta\eta_{ab} = h_{ab}$ of the longitudinal part of the metric[8]. From the Einstein equations, h_{ab} satisfies a wave equation of the form [21]:

$$\Box h_{ab} = \frac{1}{\sqrt{-g}}\,\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu h_{ab}) = 0 \ . \tag{31}$$

Following [22], we change coordinates:

$$\mathrm{d}v = \ell\, Z^{-\frac{(D-1)}{2(D-2)}}\, e^{\frac{\xi}{2a_0}(\beta-Z^2)}\,\mathrm{d}z \ , \tag{32}$$

$$\mathrm{d}s^2 = A(v)\eta_{ab}\mathrm{d}x^a\mathrm{d}x^b + A(v)\mathrm{d}v^2 + B(v)\ell^2\mathrm{d}\theta^2 \ , \tag{33}$$

and use the following Ansatz

$$h_{ab} = \epsilon_{ab}e^{ip\cdot x}e^{in\theta}\,\frac{\phi}{\psi_0} \ , \tag{34}$$

dictated by the isometries of the metric and where

$$\psi_0 := \sqrt{A^{-1}\sqrt{-g}} = \sqrt{A^{\frac{D-3}{2}}B^{\frac{1}{2}}} = Z^{\frac{D-3}{4(D-2)}}\, e^{\frac{\xi}{4a_0}(\beta-Z^2)} \ . \tag{35}$$

With these variables [1, 2, 22], Eq. (31) becomes a Schrödinger-like equation:

$$-\phi'' + \Big(\frac{\psi_0''}{\psi_0} + \frac{A}{B}n^2\Big)\phi = m^2\phi \ . \tag{36}$$

For simplicity, set $n = 0$. Integrating Eq. (32) gives

$$v - v_0 = \mathrm{sign}(z)v_*\Big[\gamma_0^{-1}\gamma\Big(\frac{D-3}{4(D-2)};\frac{\xi Z(z)^2}{2a_0}\Big) - 1\Big] \ ,$$

$$v_* = \frac{\ell}{2a_0}\,e^{\frac{\beta\xi}{2a_0}}\Big(\frac{2a_0}{\xi}\Big)^{\frac{D-3}{4(D-2)}}\gamma_0 \ , \qquad \gamma_0 = \gamma\Big(\frac{D-3}{4(D-2)};\frac{\xi}{2a_0}\Big) \ . \tag{37}$$

The change of variables $z{\to}v$ is single-valued, continuous and smooth across $z{=}0$, and $\mathrm{sign}(z) = \mathrm{sign}(v{-}v_0)$. However, the appearance of the 'incomplete gamma function.' $\gamma(a;x)$ prevents an *explicit* inversion of $v{=}v(z)$, and evaluation of ψ_0''/ψ_0 in Eq. (36). Nevertheless, in the $\xi \to 0$ limit:

$$\tilde{v} - v_0 \ = \ \mathrm{sign}(z)\,\tilde{v}_*\Big[Z(z)^{\frac{D-3}{2(D-2)}} - 1\Big] \ , \qquad \tilde{v}_* := \frac{2(D-2)}{D-3}\frac{\ell}{a_0} \ , \tag{38}$$

[8] Owing to the dearth of solutions in closed form, the analogous discussion of the de Sitter case (17–18), (22–23) is technically rather more involved, albeit just as straightforward conceptually.

which is easy to invert *explicitly*. Hereafter, we set v_0=0, focus on small but nonzero $|\xi|$ and drop the tilde. (Equivalently, we could consider the case in which $a_0, \xi > 0$ and expand around $v_\infty = \lim_{|z|\to\infty} v$, where the result is exactly the same as in the situation considered here [4].)

Eq. (36) can now be written as

$$-\left[\frac{\mathrm{d}^2}{\mathrm{d}^2 v} + \frac{\mathrm{sign}^2(v)}{4(|v_*| - |v|)^2} + \frac{1}{|v_*|}\delta(v)\right]\phi_m = m^2\phi_m \ . \tag{39}$$

Away from v=0, this becomes the Bessel equation, so

$$\phi_m = a_m\sqrt{|v_*| - |v|}\, J_0\Big(m(|v_*|-|v|)\Big) + b_m\sqrt{|v_*| - |v|}\, Y_0\Big(m(|v_*|-|v|)\Big) \ , \tag{40}$$

which must satisfy the δ-function matching conditions at v=0:

$$\left[2\frac{d\phi_m}{dv} + \frac{1}{|v_*|}\phi_m\right]_{v=0} = 0 \ . \tag{41}$$

Evaluating Eq. (40) for small values of m,

$$\phi_m \sim a_m\sqrt{|v_*| - |v|}\, 1 + b_m\sqrt{|v_*| - |v|}\, 2\,\log\left(\frac{1}{2}m(|v_*| - |v|)\right) \ , \tag{42}$$

it is clear that Eq. (41) is satisfied only if $b_m = 0$.

It remains to determine a_m such that the normalization integral of ϕ_m is m-independent

$$\langle\phi_m|\phi_m\rangle = a_m^2\int_{-v_*}^{v_*} \mathrm{d}v\ (|v_*| - |v|)J_0^2(m(|v_*| - |v|)). \tag{43}$$

This integral can in fact be computed exactly, and turns out to be dominated by the plane wave approximation, *i.e.*,

$$J_0^2(m|v_*|) \sim \frac{\cos^2(m|v_*| - \pi/4)}{m|v_*|}, \qquad m|v_*| \gg 1 \ . \tag{44}$$

and is “regularized” by $|\xi| > 0$ [4]. The zero-mode wave function can be expressed in terms of v, and the normalization integral for ψ_0 becomes:

$$\langle\psi_0|\psi_0\rangle = 2\pi\int \mathrm{d}v\ |\psi_0|^2 = \frac{2(D-2)}{D-3}\,\frac{2\pi\ell^2}{|a_0|}\,e^{\frac{\beta|\xi|}{|a_0|}} \ . \tag{45}$$

When we compare this expression with the exact result (28), the only discrepancy occurs in the power of a_0/ξ and the overall $O(1)$ numerical factor. Similar arguments apply for the ϕ_m when $\xi \neq 0$. For the normalization $\langle\phi_m|\phi_m\rangle$ we get,

$$\langle\phi_m|\phi_m\rangle_{\xi\neq 0} = e^{\frac{\beta|\xi|}{|a_0|}}\langle\phi_m|\phi_m\rangle_{\xi=0}. \tag{46}$$

The plane wave approximation is valid for $\xi \neq 0$ because $v_* \sim \ell/|a_0| \exp(\frac{\beta|\xi|}{|a_0|})$ and hence when m is large, $mv_* \gg 1$. Since large m's are limited by ℓ^{-1}, we can compute $\langle\phi_m|\phi_m\rangle$ by looking at large mv_* for which the Bessel function, J_0, looks like a plane wave (44). This means that $\langle\phi_m|\phi_m\rangle \sim a_m^2 m^{-1} v_*$ and we have to choose $a_m \sim \sqrt{m}$. Since $v_* \sim \ell$, then $\phi_m \sim \sqrt{m\ell}$.

Thus, $\psi_0 \neq \lim_{m\to 0} \phi_m$, *i.e.*, the *non-trivial stringy moduli guarantee localized gravity at* z=0 through the existence of the isolated zero mode.

With these, the Newton potential takes the following form [1, 2, 4]:

$$U(r) = \frac{1}{M_D^{D-2}} \frac{M_1 M_2}{r} \Big(1 + \frac{\ell^3}{r^3} + \cdots \Big) , \qquad (47)$$

where the correction term does not depend on a_0, β or ξ, and is very small. For example, $M_D \sim TeV$, since $\ell \sim (M_D)^{-1}$. The Newton potential has only been checked down to $r_e \sim 1\,\text{mm} \sim 10^{-12} GeV^{-1}$, so that $\ell/r < \ell/r_e \sim 10^{-15}$.

3 Dynamical Decoupling From, or Trapping Of Bulk-Roaming Modes

In addition to the degrees of freedom discussed above, typical higher-dimensional models also include degrees of freedom of various spatial extendedness (many of which describable as D-brane probes) and Yang-Mills gauge fields. For the latter, we assume that a variation of the argument shown above for gravity will similarly localize the Coulomb forces, and it remains to discuss bulk-roaming D-brane probes.

In any brane-world cosmological model, 'matter' degrees of freedom that are not localized to the brane-world through a 'topological' mechanism [17], inevitably are permitted to roam the higher-dimensional bulk of the spacetime. Since the brane-world is embedded in the bulk spacetime, this bulk-roaming matter will pass through the brane-world. Unless its interactions with *all* of the brane-world matter and *all* localized gauge fields (including gravity) are for some reason negligibly small, this will violate brane-world conservation laws. Surprisingly, our family of toy models includes an automatic dynamical mechanism for 'stabilization' in this respect.

Refs. [2, 3] have analyzed the dynamics of D-brane probes in the vicinity of the naked singularities using the appropriate Born-Infeld action [23, 24, 25, 26, 27]:

$$S_{\text{BI}} = 2\pi(2\pi\sqrt{\alpha'})^{-(p+1)} \int \mathrm{d}^{p+1}x \, \Big[C_{p+1} - e^{-\Phi}\sqrt{-\det G^s_{ab}}\Big] , \qquad (48)$$

where G^s_{ab} is the metric on the brane-probe (of p-dimensional spatial extent) induced from the background string frame metric by embedding the brane coordinates along the spacetime ones. C_{p+1} is the potential whose field strength is dual to $F := \mathrm{d}a$, where a is the axion. The induced string frame metric on the brane-probe is (v is now the speed of the brane-probe!)

$$[G^s_{ab}] \;=\; e^{\Phi/2} g_s^{-1/2} \mathrm{diag}[\,(e^{2B}v^2 - e^{2A})\,,\, \underbrace{e^{2A}, \cdots, e^{2A}}_{p}\,]\ , \tag{49}$$

whereby the action (48) may be formally identified with that of a relativistic particle, in which the rôle of 'mass' and 'speed of light' are played by rather complicated functions of the dilaton, Φ, and the metric warp factors A, B.

Unlike the supersymmetric case [27] where the effective potential (the negative of the Lagrangian evaluated at $v{=}0$) vanishes, in our case $V_{\rm eff}$ turns out to be a linear function of $Z(|z|)$ [2, 3]: another consequence of supersymmetry breaking. In the two subfamilies of toy models described in Sec. 1.2, this potential has the form depicted in Fig. 1.

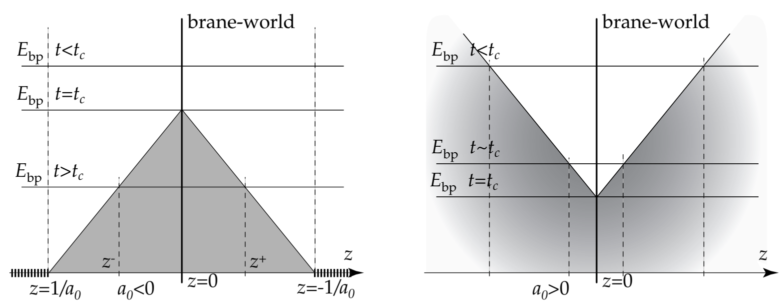

Fig. 1. The two scenarios of Sec. 1.2: On the left, naked singularities encircle the brane-world and the effective potential for the brane-probes falls off linearly. Thus, as a brane-probe looses its energy through *gravischtrahlung*, the region of the transversal plane with the brane-world becomes inaccessible to it: it *dynamically decouples from the brane-world*. On the right, the transversal plane extends to $z = \pm\infty$, and the effective potential for the brane-probe rises linearly. Now, as a brane-probe looses its energy, it becomes confined to a diminishing extent of the transversal plane, and eventually becomes *dynamically trapped to the brane-world*.

Thus, *owing to supersymmetry breaking caused by the anisotropy of the axion-dilaton system, brane-probes either decouple (when* $a_0 < 0$*) from the brane-world, or become trapped (when* $a_0 > 0$*) in it.*

In the first case, all bulk-roaming modes *eventually* decouple from the brane-world at $z = 0$. In the second, all modes *eventually* become trapped, *i.e.*, localized to the brane-world. In fact, it is amusing to realize that the latter process of localization would, from the point of view of a brane-world observer, seem as *creation of matter from nothing*—indeed, conceivably, of all of the brane-world matter.

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