Evidence for (Infinitely Diverse) Non-Convex Mirrors

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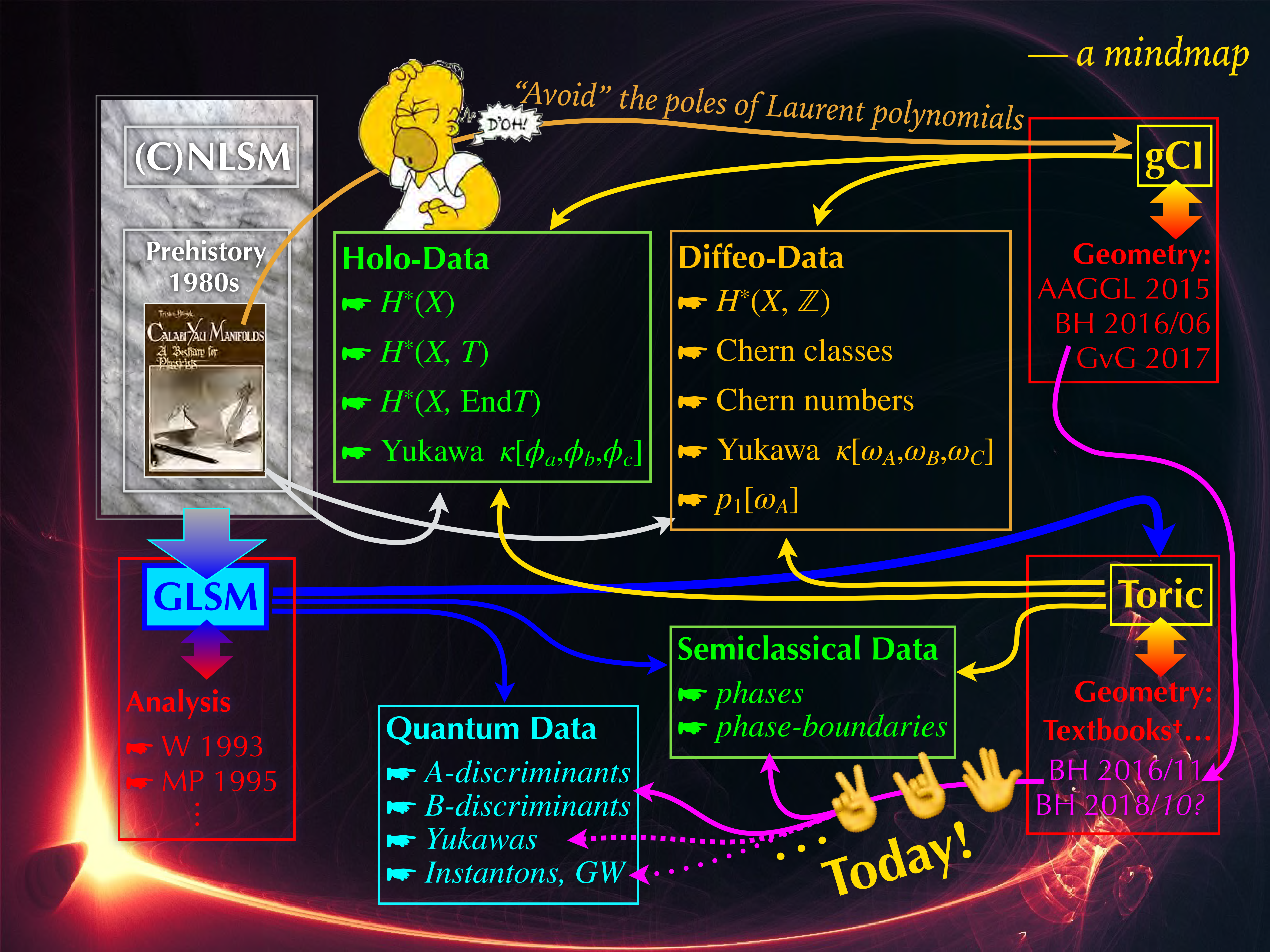

— a mindmap
"Avoid" the poles of Laurent polynomials
D'OH!
(C)NLSM
Prehistory
1980s
Calabi-Yau Manifolds
A Bestiary for Physicists
gCI
Geometry:
AAGGL 2015
BH 2016/06
GvG 2017
Holo-Data
☛ $H^*(X)$
☛ $H^*(X, T)$
☛ $H^*(X, \mathrm{End}T)$
☛ Yukawa $\kappa[\phi_a,\phi_b,\phi_c]$
Diffeo-Data
☛ $H^*(X, \mathbb{Z})$
☛ Chern classes
☛ Chern numbers
☛ Yukawa $\kappa[\omega_A,\omega_B,\omega_C]$
☛ $p_1[\omega_A]$
GLSM
Analysis
☛ W 1993
☛ MP 1995
⋮
Toric
Geometry:
Textbooks†…
BH 2016/11
BH 2018/10?
Semiclassical Data
☛ phases
☛ phase-boundaries
Quantum Data
☛ A-discriminants
☛ B-discriminants
☛ Yukawas
☛ Instantons, GW
. . . Today!

Non-Convex Mirror-Models

Prehistoric Prelude

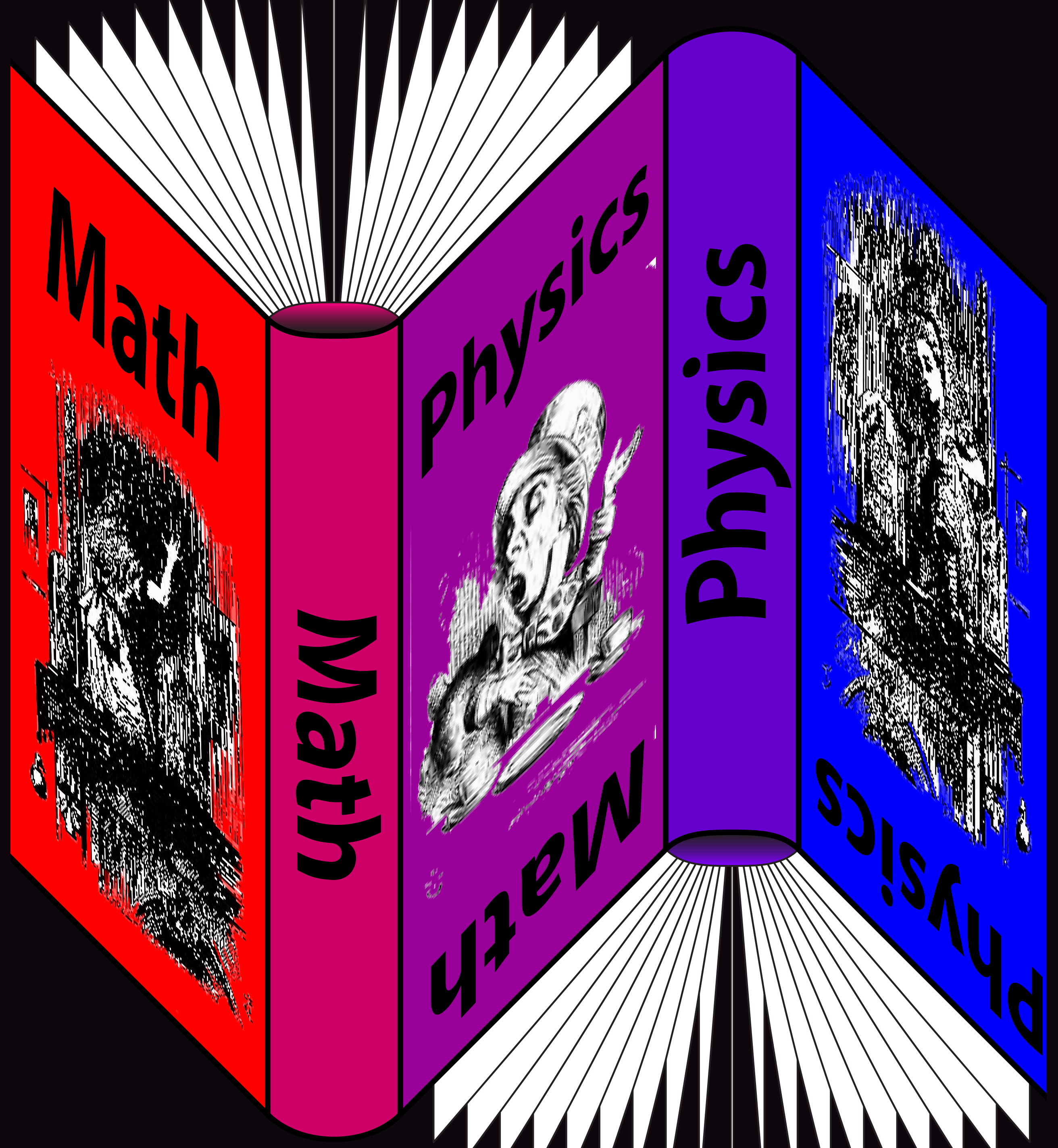
Two-Part Invention

Laurent GLSM Fugue

Discriminant Divertimento

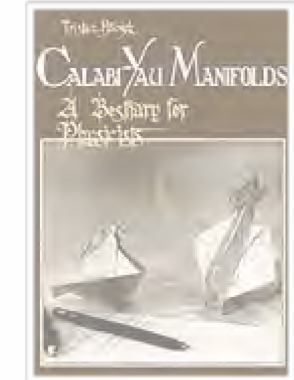
& a few Mirror Motets

"It doesn't matter what it's called,
...if it has substance."
S.-T. Yau



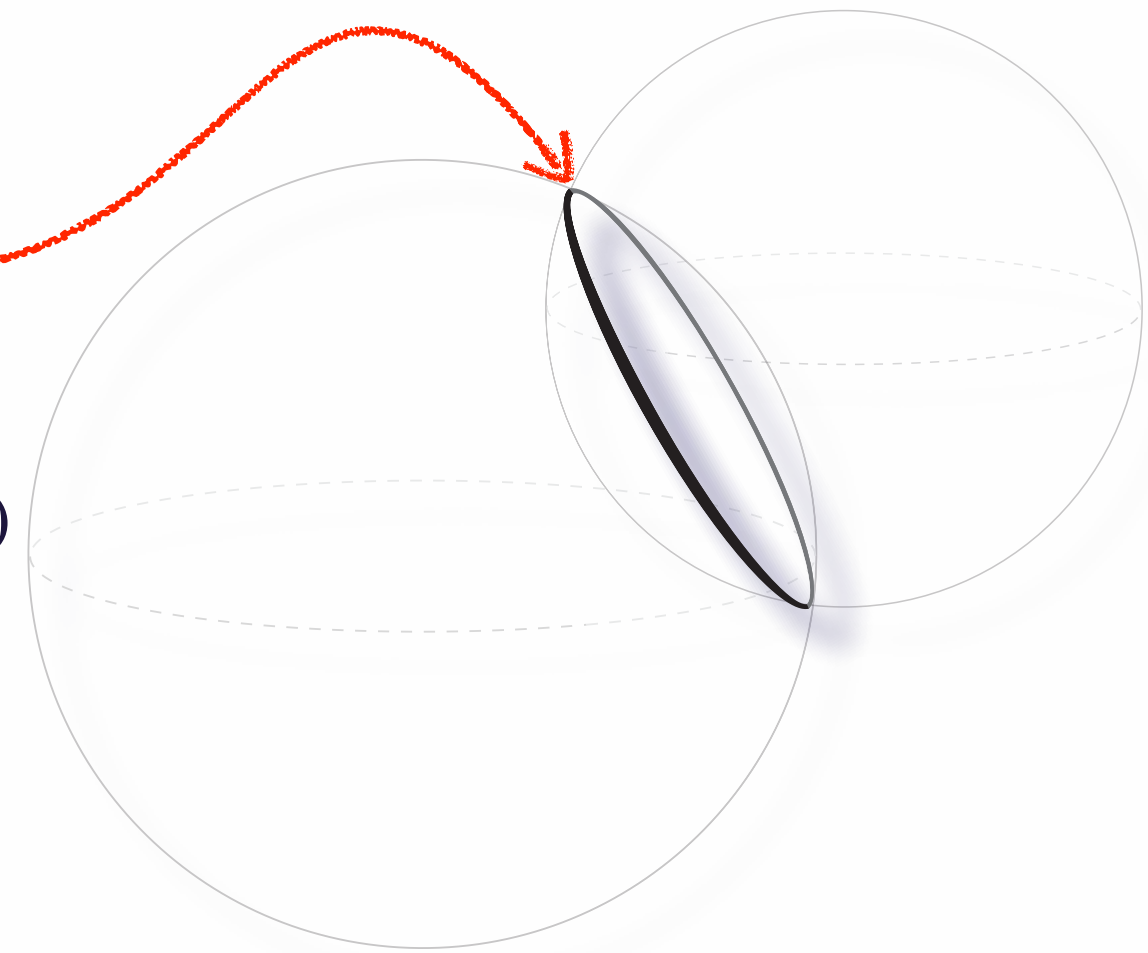
Pre-Historic Prelude
(Where are We Coming From?)

Pre-Historic Prelude



Classical Constructions

- Complete Intersections
  - Ex.: $\left.\begin{aligned}(x-x_1)^2+(y-y_1)^2+(z-z_1)^2 &= R_1{}^2\\ (x-x_2)^2+(y-y_2)^2+(z-z_2)^2 &= R_2{}^2\end{aligned}\right\}$
    - Algebraic (constraint) equations
    - …in a well-understood "ambient" (A)



- Work over complex numbers
  - …& incl. "infinity" (e.g., $\mathbb{CP}^n$'s)
- For *hypersurfaces*: $X = \{p(x) = 0\} \subset A$
  - Sections: $[f(x)]_X = [f(x) \simeq f(x) + \lambda{\cdot}p(x)]_A$
  - Differentials: $[\mathrm{d}x]_X = [\mathrm{d}x \simeq \mathrm{d}x + \lambda{\cdot}\mathrm{d}p(x)]_A$
  - Homogeneity: $\mathbb{CP}^n = \mathrm{U}(n+1)/[\mathrm{U}(1)\times\mathrm{U}(n)]$
    - r-cohomology on $\mathbb{CP}^n \to \mathrm{U}(n+1)$-tensors

Just like gauge transformations

…with $\mathrm{U}(n)$ tensors

Meromorphic Minuet

BH
1606.07420

Why Haven't We Thought of This Before?

- Holomorphic+meromorphic systems [AAGGL: 1507.03235]
- E.g: $X_m \in \left[\begin{array}{c\|c\|c} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & m & 2-m \end{array}\right]^{(2,86)}_{-168}, \qquad m = 0, 1, 2, 3, \ldots$
  - Wall: $\kappa_{111} = 2+3m$, $\kappa_{112} = 4$, so $(aJ_1+bJ_2)^3 = [2a+3(\underline{4b+ma})]a^2$.
  - Also $p_1[aJ_1+bJ_2] = -88-12(\underline{4b+ma})$… the same "$4b+ma$"
  - Thus $X_m \approx X_{m+4\gamma}$ for $\gamma \in \mathbb{Z}$: ***4 diffeo classes in the sequence***
- Are there deg(4,–1) holomorphic sections?!
  - Not on $\mathbb{P}^4 \times \mathbb{P}^1$,
  - but ***yes*** on F_m

$m=3:$	$\mathcal{O}_A\binom{3}{-4}$	$\xhookrightarrow{p}$ $\mathcal{O}_A\binom{4}{\cancel{-1}}$	$\overset{\rho_F}{\twoheadrightarrow}$ $\mathcal{O}_A\binom{4}{-1}\big\|_{F_m}$
0.	0	0	$H^0(F_m, \mathcal{Q})$
1.	$\{\varphi^{i_1(i_2i_3i_4)}_{(abc)}\}$	0	~~$H^1(F_m, \mathcal{Q})$~~
2.	0	0	~~$H^2(F_m, \mathcal{Q})$~~
⋮	⋮	⋮	⋮

$\varphi^{i(jk_1\cdots k_M)} \approx \varepsilon^{i(j}\varphi^{k_1\cdots k_M)}$, as $U(n+1) \sim U(1)\times SU(n)$ irrep.

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…in well-tempered counterpoint

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- For $\Big\{ \underbrace{x_0\, y_0^{\,m} + x_1\, y_1^{\,m}}_{:=\,\mathring{p}(x,y)} = -\sum_\alpha \epsilon_\alpha\, \delta p_\alpha(x,y) \Big\} = F^{(n)}_{m;\epsilon} \in \left[\begin{matrix}\mathbb{P}^n \\ \mathbb{P}^1\end{matrix}\right\Vert\left.\begin{matrix}1 \\ m\end{matrix}\right]$ *F. Hirzebruch, 1951*
  - Lefschetz hyperplane thm: $H^r(F^{(n)}_{m;\epsilon}, \mathbb{Z}) \approx H^r(\mathbb{P}^n \times \mathbb{P}^1, \mathbb{Z})$, for $r < n$,
  - …and also for $r = n$ (& then $r > n$) since $\chi_E(F^{(n)}_{m;\epsilon}) = 2n$.
  - Chern class:

$$c(F^{(n)}_{m;p}) = \left[\frac{(1+J_1)^{n+1}(1+J_2)^2}{(1+J_1+m\,J_2)}\right]_{\substack{J_1^{\,n+1}=0\\ J_2^{\,2}=0}} \qquad \frac{(1+J_2)^2}{(1+J_1+mJ_2)} = 1+J_1-mJ_2$$

  - …so that:

$$= (1+J_1)^{n-1}(1+J_2)^2(1+J_1-mJ_2) = \prod_i \Big(1 + \sum_a Q^a(x_i)J_a\Big)$$

- For $(n=4)$:

$$C_1^{\,4} = 512, \quad C_1^{\,2}{\cdot}C_2 = 224, \quad C_1{\cdot}C_3 = 56, \quad C_2^{\,2} = 96, \quad C_4 = \chi_E = 8,$$

$$C_1^{\,3}[aJ_1+bJ_2] = 16[6a + (\underline{4b+am})], \qquad C_1{\cdot}C_2[aJ_1+bJ_2] = 2[22a + 3(\underline{4b+am})],$$

$$C_3[aJ_1+bJ_2] = 12a + (\underline{4b+am}),$$

$$C_1^{\,2}[(aJ_1+bJ_2)^2] = 8a[2a + (\underline{4b+am})], \qquad C_2[(aJ_1+bJ_2)^2] = a(8a + 3(\underline{4b+am})],$$

$$C_1[(aJ_1+bJ_2)^3] = a^2(2a + 3(\underline{4b+am})], \qquad [(aJ_1+bJ_2)^4]_{F_m} = a^3(\underline{4b+am}).$$

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- For $\left\{ \underbrace{x_0\, y_0^{\,m} + x_1\, y_1^{\,m}}_{:= \mathring{p}(x,y)} = -\sum_\alpha \epsilon_\alpha\, \delta p_\alpha(x,y) \right\} = F^{(n)}_{m;\epsilon} \in \left[\begin{matrix}\mathbb{P}^n\\ \mathbb{P}^1\end{matrix} \middle\| \begin{matrix}1\\ m\end{matrix}\right]$
  - The central $(\epsilon = 0)$ member of the family has all the requisite features:
  - Directrix: $S := \{\mathfrak{s}(x,y) = 0\}$, $[S] = [H_1] - m[H_2]$ & $S^n = -(n-1)m$;
  - Extra anticanonicals: $\dim H^0(F^{(n)}_{m;\epsilon}, \mathcal{K}^*) = 3\binom{2n-1}{n} + \delta_{\epsilon,0} \cdot \vartheta^m_3 \cdot \binom{2n-2}{2}(m-3)$
  - Extra T-bundle valued: $\dim H^0(F^{(n)}_{m;\epsilon}, T) = n^2 + 2 + \delta_{\epsilon,0} \cdot \vartheta^m_1 \cdot (n-1)(m-1)$

$$\chi(\mathcal{K}^{\otimes k}_{F^{(n)}_{m;\epsilon}}) := \sum_{i=0}^{n} (-1)^i \dim H^i(F^{(n)}_{m;\epsilon}, \mathcal{K}^{\otimes k}) = \frac{(1-2k)}{(n-1)!}(1+nk) \prod_{j=1}^{n-2} \Big(n(k+1) - j \Big)$$

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- For $\{\underbrace{x_0\, y_0^{\,m} + x_1\, y_1^{\,m}}_{:=\mathring{p}(x,y)} = -\sum_\alpha \epsilon_\alpha\, \delta p_\alpha(x,y)\} = F^{(n)}_{m;\epsilon} \in \left[\begin{matrix}\mathbb{P}^n\\ \mathbb{P}^1\end{matrix}\middle\|\begin{matrix}1\\ m\end{matrix}\right]$ ($\epsilon_\alpha \to 0$)
  - The central ($\epsilon = 0$) member of the family has all the requisite features:
  - Directrix: $S := \{\mathfrak{s}(x,y) = 0\}$, $[S] = [H_1] - m[H_2]$ & $S^n = -(n-1)m$;
  - Extra anticanonicals: $\dim H^0(F^{(n)}_{m;\epsilon}, \mathcal{K}^*) = 3\binom{2n-1}{n} + \delta_{\epsilon,0}\cdot\vartheta^m_3\cdot\binom{2n-2}{2}(m-3)$
  - Extra T-bundle valued: $\dim H^0(F^{(n)}_{m;\epsilon}, T) = n^2+2 + \delta_{\epsilon,0}\cdot\vartheta^m_1\cdot(n-1)(m-1)$
  - …exactly as computed for $F^{(n)}_m := \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(m)^{\oplus(n-1)}\big)$ "Linear algebra"
  - All "extras" are lost for generic ($\epsilon_\alpha \neq 0$) deformations, resulting in the ***discrete*** deformation $F^{(n)}_m \to F^{(n)}_{m\ (\mathrm{mod}\ n)}$.

- Also, explicit tensorial (residue) representatives → can compute coupling ratios

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- E.g: $X_m \in \left[\begin{array}{c||c|c} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & m & 2-m \end{array}\right] \subset F_m \in \left[\begin{array}{c||c} \mathbb{P}^4 & 1 \\ \mathbb{P}^1 & m \end{array}\right] = \{ p_{a(j_1\cdots j_m)}\, x^a\, y^{j_1}\cdots y^{j_m} = 0 \}$
  - F_m: at $z \in \mathbb{P}^1$, $\mathbb{P}^4[1] = \mathbb{P}^3$; so F_m is a deg-m fibration of $\mathbb{P}^3$ over $\mathbb{P}^1$.
  - X_m is an anticanonical (CY) hypersurface in F_m.
- Cohomology maps made explicit (m=3):

$m=3$:	$\mathcal{O}_A\binom{3}{-4}$ $\overset{p}{\hookrightarrow}$	$\mathcal{O}_A\binom{4}{-1}$ $\overset{\rho_F}{\twoheadrightarrow}$	$\mathcal{O}_A\binom{4}{-1}\vert_{F_m}$
0.	0	0	$H^0(F_m, \mathcal{Q})$
1.	$\{\varphi^{i_1(i_2 i_3 i_4)}_{(abc)}\}$	0	~~$H^1(F_m, \mathcal{Q})$~~
2.	0	0	~~$H^2(F_m, \mathcal{Q})$~~
⋮	⋮	⋮	⋮

$\varphi^{i(jk_1\cdots k_M)} \approx \varepsilon^{i(j}\varphi^{k_1\cdots k_M)}$, as $\mathrm{U}(n{+}1) \sim \mathrm{U}(1)\times\mathrm{SU}(n{+}1)$

Source: H^q for codim $= q+1$ CYn-fold

$$q^i_{(abcd)} := f^{i(jkl)}_{(abc} \cdot p_{d)(jkl)}$$

q-maps now factor thru *p*-maps!

No longer independent in the Koszul resolution for X_m!

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- E.g: $X_m \in \left[\begin{matrix}\mathbb{P}^4 \\ \mathbb{P}^1\end{matrix}\middle\|\begin{matrix}1 \\ m\end{matrix}\middle|\begin{matrix}4 \\ 2-m\end{matrix}\right] \subset F_m \in \left[\begin{matrix}\mathbb{P}^4 \\ \mathbb{P}^1\end{matrix}\middle\|\begin{matrix}1 \\ m\end{matrix}\right] = \{p_{a(j_1\cdots j_m)}\, x^a\, y^{j_1}\cdots y^{j_m} = 0\}$

- To be precise:

	$\mathcal{O}_A\binom{3}{2-2m}$	$\overset{p}{\hookrightarrow}$	$\mathcal{Q} = \mathcal{O}_A\binom{4}{2-m}$	$\overset{\rho_F}{\twoheadrightarrow}$	$\mathcal{Q}\|_{F_m}$	
0.	$\theta^1_m\{\varphi_{(abc)(i_1\cdots i_{2-2m})}\}$	$\overset{p}{\to}$	$\theta^2_m\{\phi_{(abcd)(i_1\cdots i_{2-m})}\}$	$\overset{\rho_F}{\longrightarrow}$	$H^0(F_m,\mathcal{Q})$	$\overset{\mathrm{d}}{\to}$
1.	$\theta^m_2\{\varepsilon^{i(j}\varphi^{k_1\cdots k_{2m-4})}_{(abc)}\}$	$\overset{p}{\to}$	$\theta^m_4\{\varepsilon^{i(j}\phi^{k_1\cdots k_{m-4})}_{(abcd)}\}$	$\overset{\rho_F}{\longrightarrow}$	$H^1(F_m,\mathcal{Q})$	$\overset{\mathrm{d}}{\to}$
2.	0		0		$H^2(F_m,\mathcal{Q}) = 0$	
$\vdots$	$\vdots$		$\vdots$		$\vdots$	

$$\theta^n_m = \begin{cases} 1 & m \leqslant n, \\ 0 & m > n. \end{cases}$$

"Linear algebra"

$$0 \to H^0(A,\mathcal{Q}) \overset{\rho_F}{\longrightarrow} H^0(F_2,\mathcal{Q}) \overset{\mathrm{d}}{\to} H^1\big(A,\mathcal{O}\big(\tbinom{3}{-2}\big)\big) \to 0,$$

$$H^0(F_2,\mathcal{Q}) = \Big\{\Big(\phi_{(abcd)} + \gamma^{\,i}_{(abcd)\,k}\,\frac{y^k}{y^j}\Big)\, x^a x^b x^c x^d\Big\}, \qquad \gamma^{\,i}_{(abcd)\,k} \overset{\text{def}}{=} \varepsilon^{ij} f_{(abc}\,p_{d)(jk)}$$

$m = 2$

$$\gamma(x,y) = \varphi(x)\left[p_{00}(x)\frac{y^0}{y^1} - p_{11}(x)\frac{y^1}{y^0} + \lambda\underbrace{\Big(p_{00}(x)\,\frac{y^0}{y^1} + 2p_{01}(x) + p_{11}(x)\,\frac{y^1}{y^0}\Big)}_{=0 \text{ on } F_2 \text{ owing to (A.11)}}\right].$$

Akin to the Wu-Yang magnetic monopole gauge transformation

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…in well-tempered counterpoint

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- E.g: $X_m \in \left[\begin{matrix}\mathbb{P}^4\\ \mathbb{P}^1\end{matrix}\,\middle\|\,\begin{matrix}1\\ m\end{matrix}\,\middle|\,\begin{matrix}4\\ 2-m\end{matrix}\right] \subset F_m \in \left[\begin{matrix}\mathbb{P}^4\\ \mathbb{P}^1\end{matrix}\,\middle\|\,\begin{matrix}1\\ m\end{matrix}\right] = \{p_{a(j_1\cdots j_m)}\, x^a\, y^{j_1}\cdots y^{j_m} = 0\}$
  - The Koszul resolution complicated by factoring $q^i_{(abcd)} := f^{i(jkl)}_{(abc}\cdot p_{d)(jkl)}$

$$\begin{array}{ccccc} & & \mathcal{O}\binom{-4}{m-2} & & \\ & \overset{p}{\nearrow} & \big\downarrow \varepsilon f & \overset{q}{\searrow} & \\ \mathcal{O}\binom{-5}{-2} & & & & \mathcal{O}_A \twoheadrightarrow \mathcal{O}_X \\ & \underset{q}{\searrow} & & \underset{p}{\nearrow} & \\ & & \mathcal{O}\binom{-1}{-m} & & \end{array} \qquad \left\{\begin{array}{l} p \in H^0(A, \mathcal{O}\binom{1}{m}) \\ \varepsilon f \leftarrow H^1(A, \mathcal{O}\binom{4}{2-m}) \\ q \in H^0(F_m, \mathcal{O}\binom{4}{2-m}) \end{array}\right.$$

  - The induced cohomology εf-map acts from $H^q \to H^{q+1}$,
  - The q-sections may be "complicated" by denominator factor

$$q(x,y) := f^{i(j_1\cdots j_{m-2}\, j_{m-1}\cdots j_{2m-3})}_{(abc}\cdot p_{d)(i\, j_{m-1}\cdots j_{2m-3})}\,\frac{x^a x^b x^c x^d}{g^{(j_1\cdots j_{m-2})}(y)}$$

  - …which serves only to spread out the poles (if so desired)
    - …the choice of which is the only factor *not* dictated by "linear algebra"
  - …also, can (and may need to) "un-contract" indices: $\delta^i_j \to (y^i/y_j)$.

New Prospects

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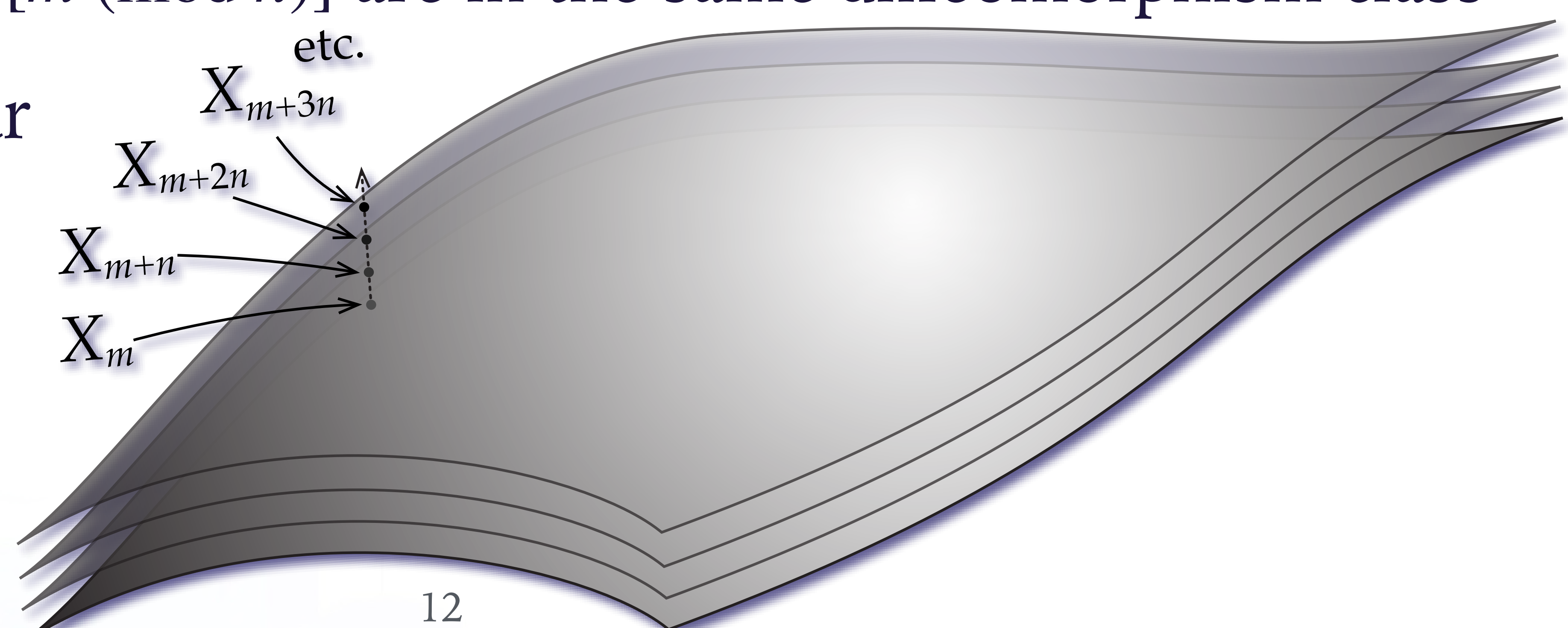
Beyond the (Theorem of) Wall

- The (mod 4) periodicity is not so crazy after all…

$$\Big\{\underbrace{x_0\,y_0^{\,m} + x_1\,y_1^{\,m}}_{:=\mathring{p}(x,y)} = -\sum_\alpha \epsilon_\alpha\,\delta p_\alpha(x,y)\Big\} = F^{(n)}_{m;\epsilon} \in \left[\begin{matrix}\mathbb{P}^n \\ \mathbb{P}^1\end{matrix}\middle\|\begin{matrix}1 \\ m\end{matrix}\right]$$

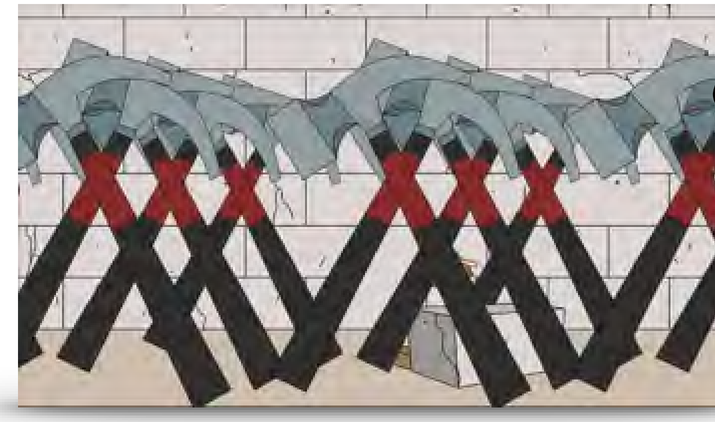
- For $\epsilon \neq 0$: $F_{m;\epsilon}$. However, for $\epsilon = 0$ this is $F_{m;0}$ (has a $C\cdot C = -m$).
- Since $F_{m;\epsilon}$ are both rigid, $\mathscr{M} = \mathbb{C}_\epsilon\,/\,$reparam. = 2 pts.
  - …but all $F_{m;\epsilon}$, for [m (mod n)] are in the same diffeomorphism class
- Something similar happens with the CY(n–1)-folds X_m



The Big Picture
(What are We Doing?)

New Prospects

Beyond the (Theorem of) Wall



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- The previous example and its cousins:

$h^{1,1} = 2,\ h^{2,1} = 86;\ \dim H^1(X_m, \operatorname{End} T) = 188$

κ_{ABC} and $p_1[\omega_A]$ vary

in a (mod 4) fashion

$$\cdots \approx \left[\begin{array}{c||c|c} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & 6 & -4 \end{array}\right] \approx \left[\begin{array}{c||cc} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & 2 & 0 \end{array}\right] \quad \leftarrow \quad \left[\begin{array}{c||cc} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & 1 & 1 \end{array}\right] \approx \left[\begin{array}{c||c|c} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & 5 & -3 \end{array}\right] \approx \cdots$$

$$\downarrow \qquad [\mathbb{P}^5 \| 2, 4] \rightleftarrows\ \downarrow \qquad \mathbb{P}^4[5] \rightleftarrows \qquad \uparrow \qquad \uparrow$$

$$\cdots \approx \left[\begin{array}{c||c|c} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & 7 & -5 \end{array}\right] \approx \left[\begin{array}{c||c|c} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & 3 & -1 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{c||cc} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & 0 & 2 \end{array}\right] \approx \left[\begin{array}{c||c|c} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & 4 & -2 \end{array}\right] \approx \cdots$$

$$\left[\begin{array}{c||c} \mathbb{P}^3 & 4 \\ \mathbb{P}^1 & 2 \end{array}\right] \rightleftarrows [\mathbb{P}^4_{(1:1:1:1:4)} \| 8] \qquad \mathbb{P}^4_{(1:1:1:1:2)}[6]$$

New Prospects

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Beyond the (Theorem of) Wall

- The previous example and its cousins:

$h^{1,1} = 2,\ h^{2,1} = 86;\ \dim H^1(X_m, \mathrm{End}\,T) = 188$

κ_{ABC} and $p_1[\omega_A]$ vary in a (mod 4) fashion

$$X_J : \mathcal{M}(\Delta_5^\circ) \leftrightarrow X_K : \widetilde{\mathcal{M}}_5(\Delta_5^\circ)$$

$$X_C : \mathcal{M}_4(\Delta_2^\circ) \qquad X_B : \mathcal{M}(\Delta_1^\circ) \not\approx X_K : \widetilde{\mathcal{M}}_5(\Delta_5^\circ)$$

$$\cdots \approx \begin{bmatrix} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & 6 & -4 \end{bmatrix} \approx \begin{bmatrix} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & 2 & 0 \end{bmatrix} \leftarrow \begin{bmatrix} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & 1 & 1 \end{bmatrix} \approx \begin{bmatrix} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & 5 & -3 \end{bmatrix} \approx \cdots$$

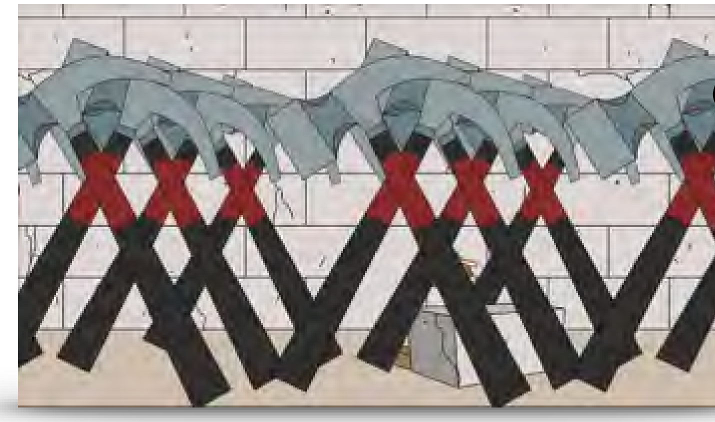
$$\downarrow \qquad \downarrow \qquad \uparrow \qquad \uparrow$$

$$\cdots \approx \begin{bmatrix} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & 7 & -5 \end{bmatrix} \approx \begin{bmatrix} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & 3 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & 0 & 2 \end{bmatrix} \approx \begin{bmatrix} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & 4 & -2 \end{bmatrix} \approx \cdots$$

$$X_E : \widetilde{\mathcal{M}}_3(\Delta_3^\circ) \qquad X_A : \mathcal{M}(\Delta_0^\circ) \not\approx X_G : \widetilde{\mathcal{M}}_4(\Delta_4^\circ)$$

$$X_D : \mathcal{M}(\Delta_3^\circ) \leftrightarrow X_E \qquad X_G \leftrightarrow X_F : \mathcal{M}(\Delta_4^\circ)$$

New Prospects



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Beyond the (Theorem of) Wall

- Segue to toric (re)incarnation:

Corollary 1.1 (toric vs. bi-projective). *The Hirzebruch n-folds defined by the central bi-projective embedding* $\big(F_m^{(n)} := \{x_0\, y_0{}^m + x_1\, y_1{}^m = 0\}\big) \subset \mathbb{P}^n \times \mathbb{P}^1$ *are isomorphic to the toric varieties specified as*

$$F_m^{(n)} \subset \left[\begin{array}{c||c} \mathbb{P}^4 & 1 \\ \mathbb{P}^1 & m \end{array}\right] \qquad \begin{array}{c|c|cccc|cc|} & v_0 & v_1 & v_2 & \cdots & v_n & v_{n+1} & v_{n+2} \\ \hline & 0 & -1 & 1 & \cdots & 0 & 0 & -m \\ \Delta^\star_{F_m} & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ & 0 & -1 & 0 & \cdots & 1 & 0 & -m \\ & 0 & 0 & 0 & \cdots & 0 & 1 & -1 \\ \hline Q^1 & -4 & 1 & 1 & \cdots & 1 & 0 & 0 \\ Q^2 & m-2 & -m & 0 & \cdots & 0 & 1 & 1 \end{array} \qquad \begin{array}{c} \textbf{Cox var's} \\ X_\rho := X_{v_\rho}, \end{array}$$

and the explicit isomorphism of homogeneous and Cox coordinates respectively is given as:

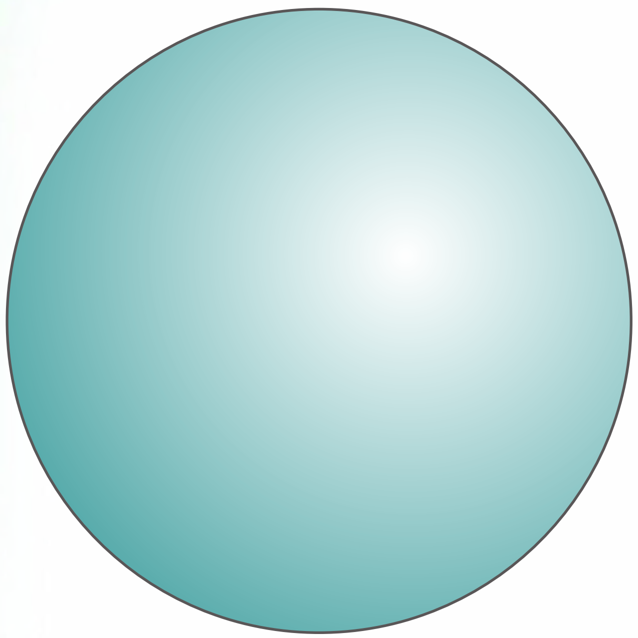
$$\mathbb{P}^n \times \mathbb{P}^1 \ni (x_0, x_1, \cdots, x_n; y_0, y_1) \mapsto \begin{cases} X_1 = \Big[\Big(\frac{x_0}{y_1{}^m} - \frac{x_1}{y_1{}^m}\Big) + \frac{\lambda}{(y_0\, y_1)^m}\mathring{p}(x,y)\Big] \\ X_2 = x_2, \ \cdots\ X_n = x_n,\ X_{n+1} = y_0,\ X_{n+2} = y_1, \end{cases}$$

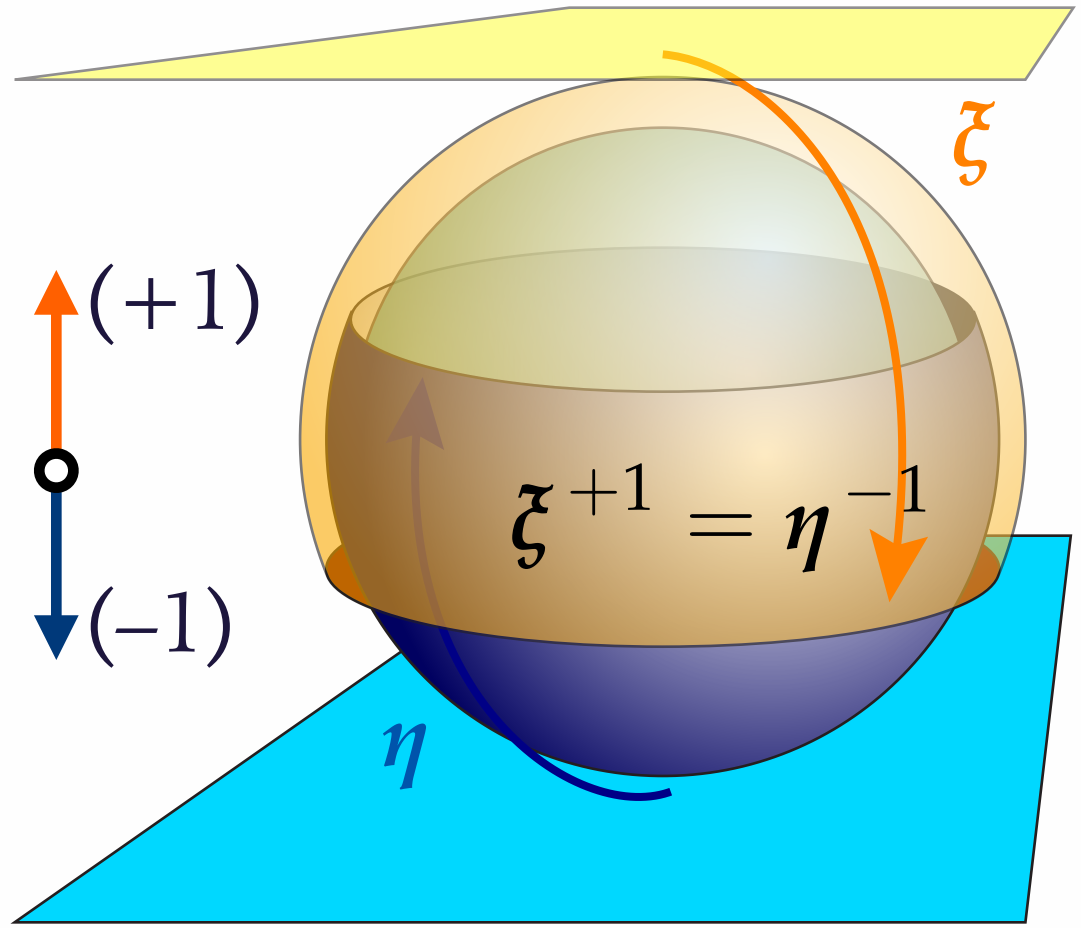
where $\big(S := \{X_1 = 0\}\big) \subset F_m^{(n)}$ *is the hallmark* ***directrix*** *[28], parametrizes the MPCP-desingularization, and has the maximally negative self-intersection* $S \cdot \ldots \cdot S = -(n-1)m$.

Two-Part Invention

Toric Geometry

- Consider $S^2 \simeq \mathbb{P}^1$:





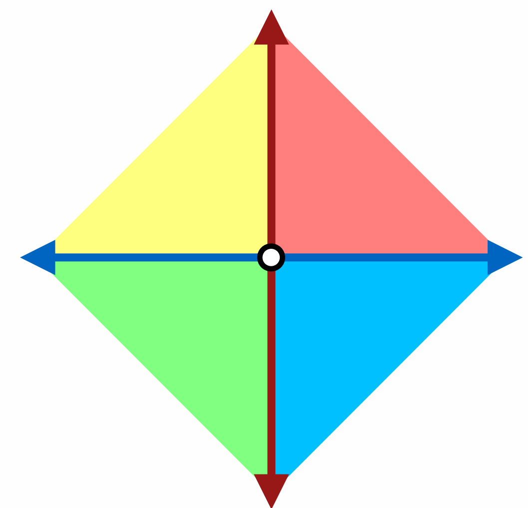
- Need at least two (complex) coordinates:
  - Match (the exponents) near the equator: $(+1)_N = (-1)_S$
  - Symmetry: $\xi \to \lambda^{+1}\xi$ and $\eta \to \lambda^{-1}\eta$, with $\lambda \in \mathbb{C}^* = (\mathbb{C} \smallsetminus \{0\})$
  - Explicitly: $\lambda = e^{i(\alpha+i\beta)} = e^{-\beta} \cdot e^{i\alpha}$ = (real) rescaling · phase-change

“thickened” S^1 *usual gauge transformation*

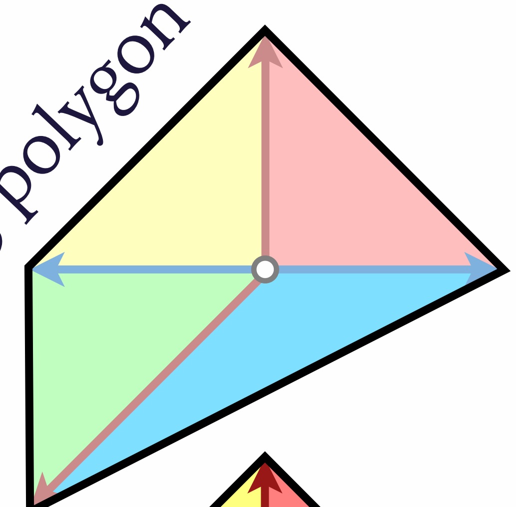
Two-Part Invention

Toric Geometry

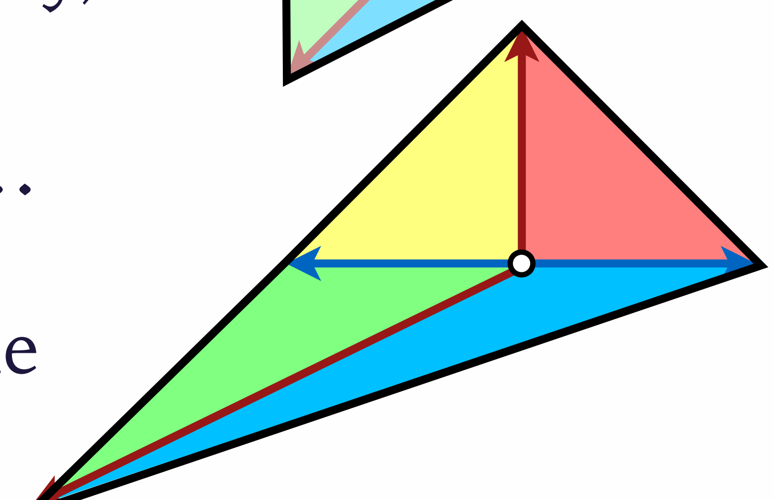
- More complicated examples: $S^2 \times S^2$
  - An entire 2nd sphere at every point of 1st
  - Orthogonal ↔ linearly independent
  - Top-dim cones ↔ coord. patches
  - 2-dim (enveloping) polytope ↔ ($\mathbb{C}$) 2-dim. geometry



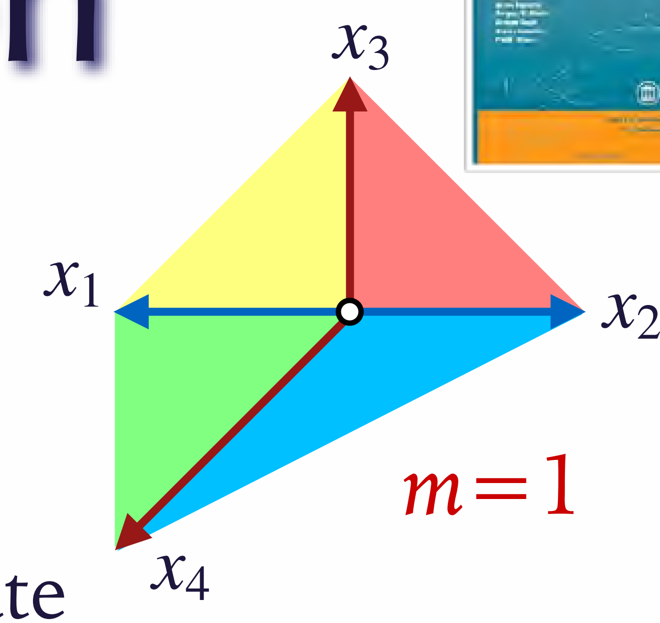
- Now: Hirzebruch ($\mathbb{C}$) surface, $\mathfrak{F}_1$. 🙂
  - "Slanting" (0,–1) → (–m,–1) the bottom vertex (& two cones) encodes the "twist"
  - … $\mathfrak{F}_m$ = m-twisted $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.
  - …and so on: 4 textbooks worth…



- …focusing exclusively on convexity…
  - wherein "cone" is *defined* to mean 😱 ***strongly convex*** rational polyhedral cone



Two-Part Invention



Fan Encoding

- The fan encodes the space
- …but also its symmetries:
  - Each primitive generator ↦ (Cox) coordinate
  - Read off cancelling relations

$$1\,\vec{v}_{x_1} + 1\,\vec{v}_{x_2} + 0\,\vec{v}_{x_3} + 0\,\vec{v}_{x_4} = 0$$

$$(x_1, x_2, x_3, x_4) \simeq (\lambda^1\, x_1, \lambda^1\, x_2, \lambda^0\, x_3, \lambda^0\, x_4)$$

$$0\,\vec{v}_{x_1} + m\,\vec{v}_{x_2} + 1\,\vec{v}_{x_3} + 1\,\vec{v}_{x_4} = 0$$

$$(x_1, x_2, x_3, x_4) \simeq (\lambda^0\, x_1, \lambda^m\, x_2, \lambda^1\, x_3, \lambda^1\, x_4)$$

- Defines two independent (gauge) symmetries
  - a GLSM w/gauge-invariant Lagrangian
  - and |*ground state*⟩ where KE = 0 = PE
  - & (quantum) Hilbert space on it

Laurent GLSM Fugue
(& *new-fangled* Toric Geometry)

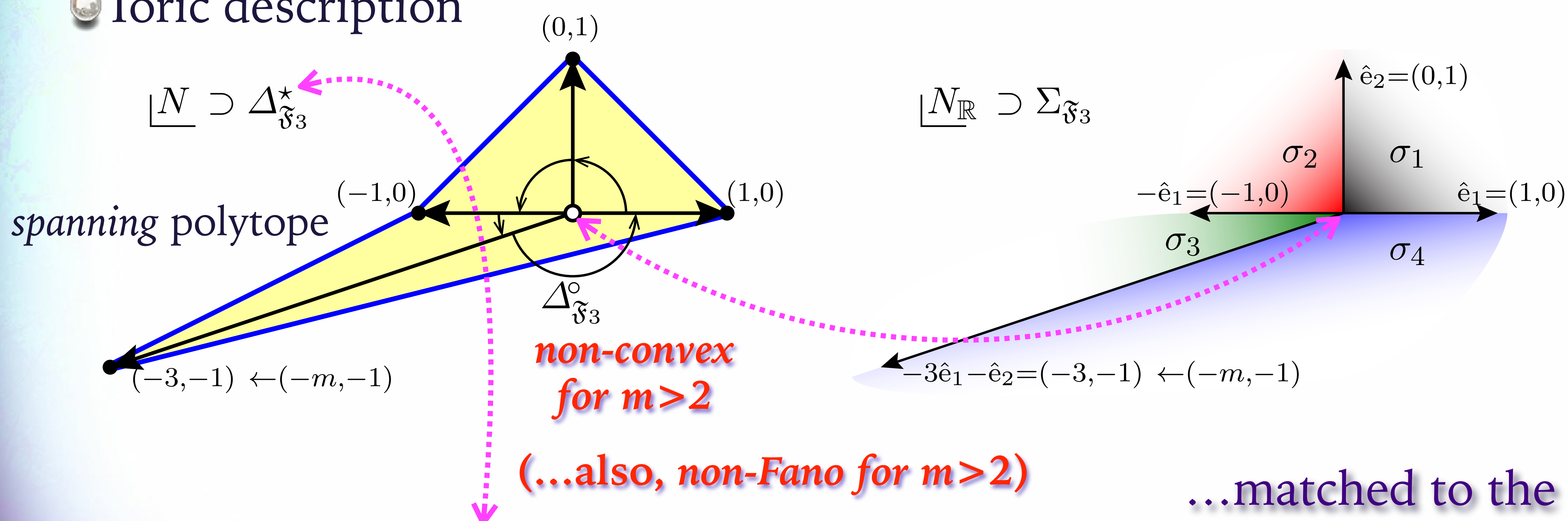
A Generalized Construction of
Calabi-Yau Models and Mirror Symmetry
arXiv:1611.10300
+ any day now…

Laurent GLSM Fugue

& Non-Convex Mirrors

—*Proof-of-Concept*—

- 2-torus in the Hirzebruch surface $\mathfrak{F}_m$:
  - "Anticanonical" (Calabi-Yau, Ricci-flat) hypersurface in $\mathfrak{F}_m$
  - Toric description



The star-triangulation of the *spanning* polytope defines the fan of the underlying toric variety

…matched to the bi-projective embedding via diffeo & holo data

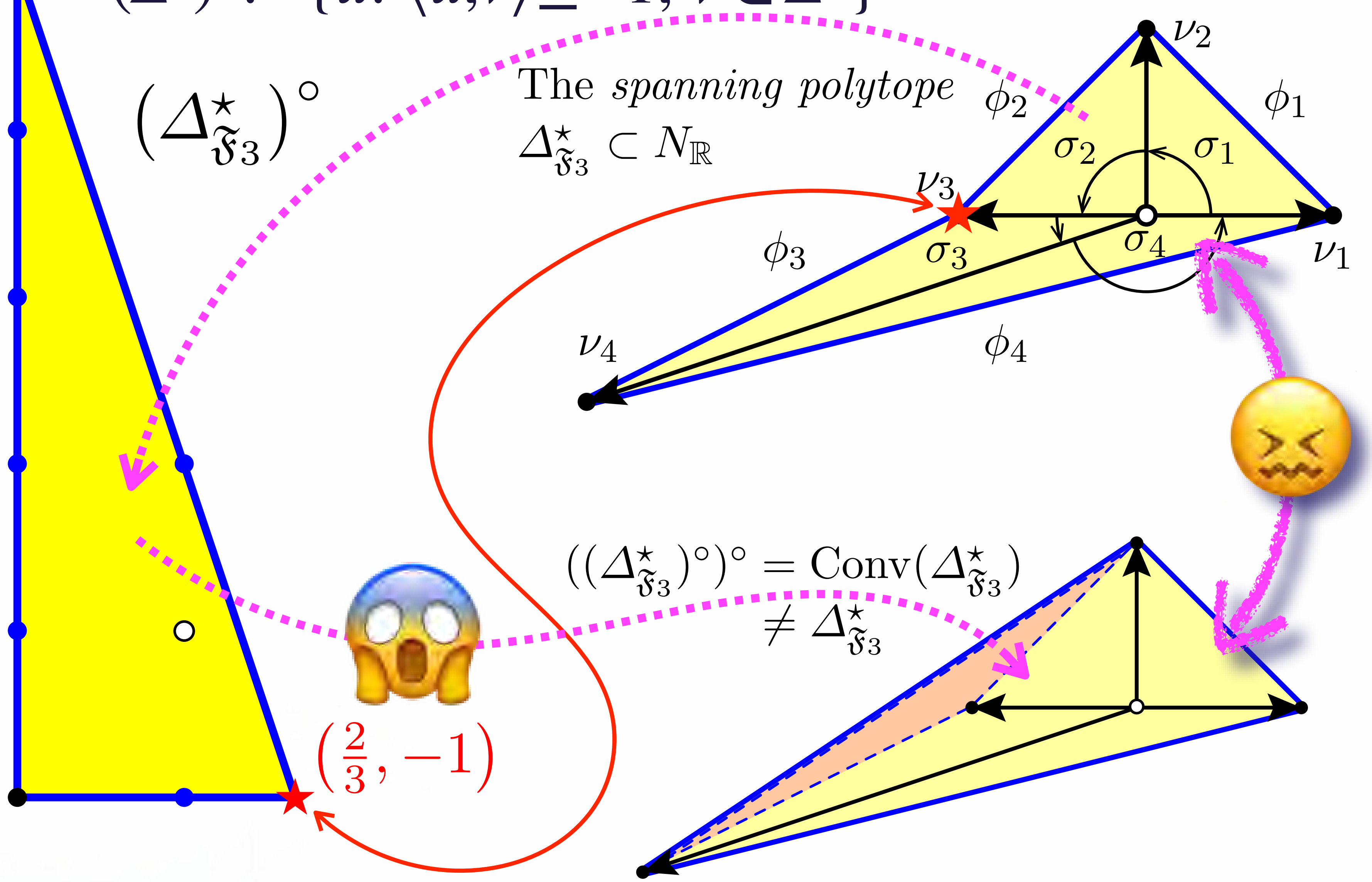
Laurent GLSM Fugue

& Non-Convex Mirrors

—*Proof-of-Concept*— arXiv:1611.10300

- The *Newton* polytope (polar of *spanning* polytope):
  - The “standard” polar polytope is non-integral
  - The “standard” polar of the polar is not the spanning polytope that we started with
  - Is no good for mirror symmetry

$(\Delta^\star)^\circ := \{u\colon \langle u,v\rangle \geq -1,\ v\in\Delta^\star\}$ ← global



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& Non-Convex Mirrors

—*Proof-of-Concept*— arXiv:1611.10300

- The ***standard*** Newton polytope:
  - specifies allowed monomials
  - The so-defined 2-tori are all ***singular*** @ (0,0,1)
  - …as each monomial has at least an x_1 factor, so $f(x) = x_1 \cdot g(x)$

There must be more to this!

$x_1^2 x_3^5$

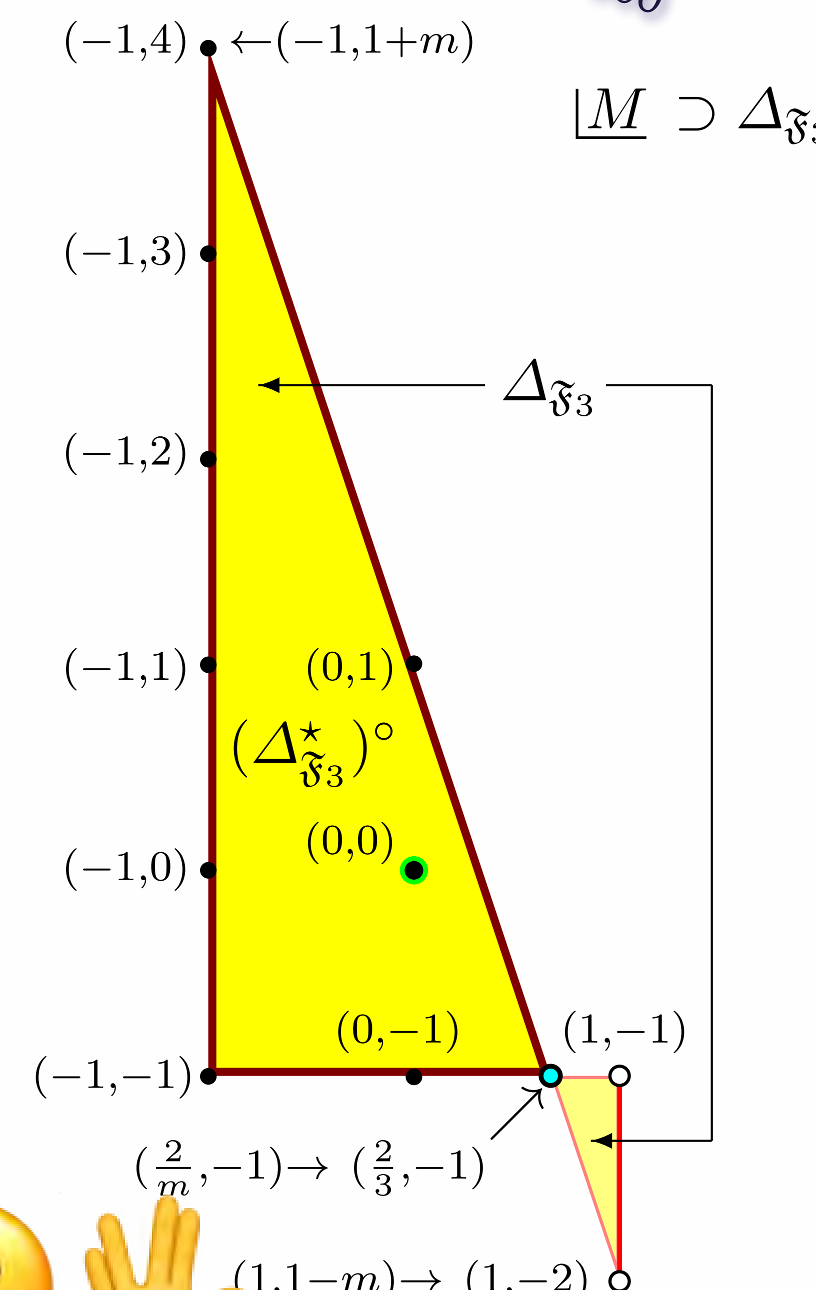
$x_1^2 x_3^4 x_4$

$x_1^2 x_3^3 x_4^2$

$x_1 x_2 x_3^2 \quad x_1^2 x_3^2 x_4^3$

$x_1 x_2 x_3 x_4 \quad x_1^2 x_3 x_4^4$

$x_1 x_2 x_4^2 \quad x_1^2 x_4^5$



- The extension corresponds to Laurent monomials:

$$\left.\begin{array}{l}(1,-1) \mapsto \dfrac{x_2^2}{x_4} \\ (1,-2) \mapsto \dfrac{x_2^2}{x_3}\end{array}\right\}$$ make the 2-tori Δ-regular*. 🙂 🖖

*→ "intrinsic limit"

Laurent GLSM Fugue

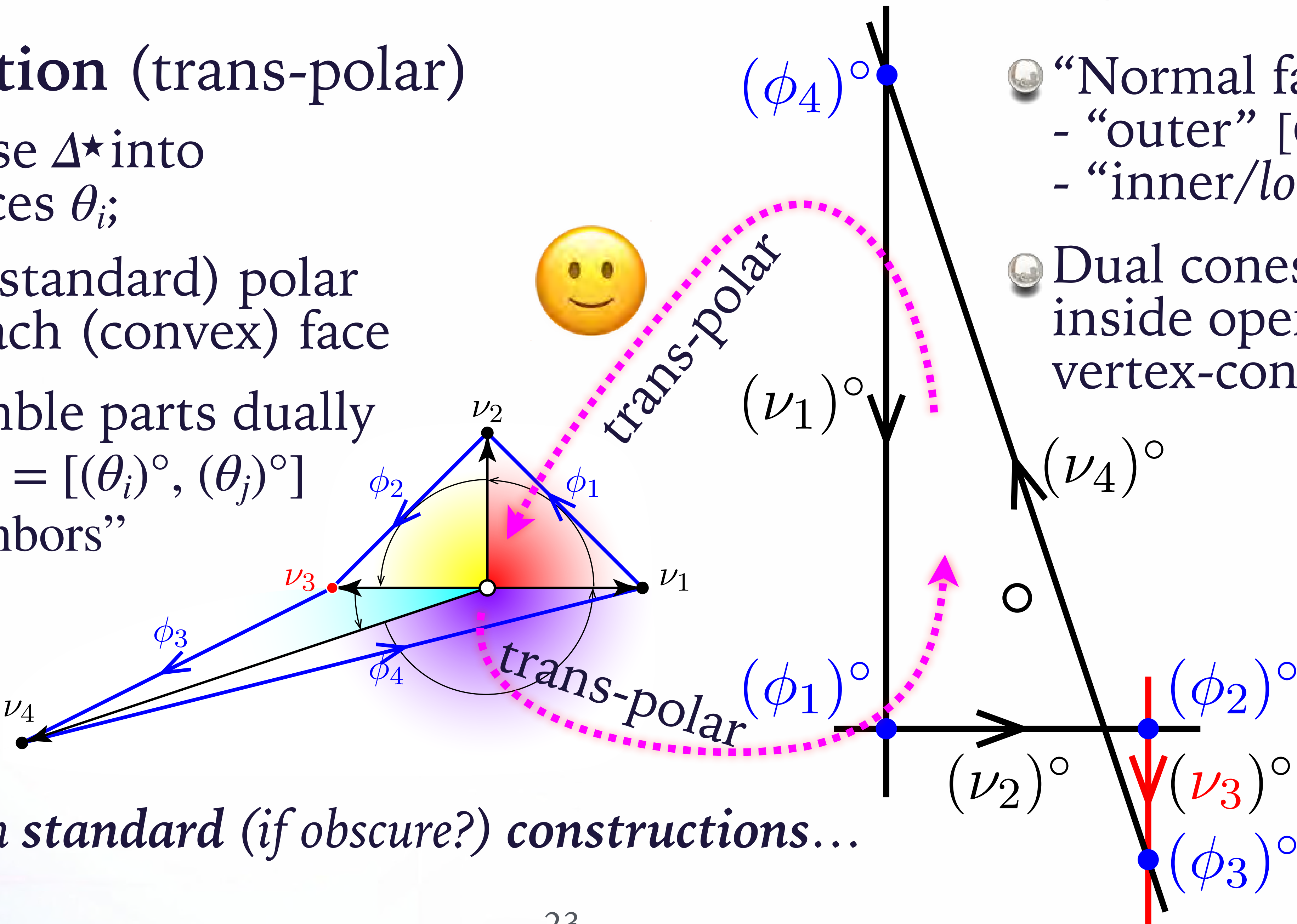
& Non-Convex Mirrors

—*Proof-of-Concept*—

- The *oriented Newton* polytope (trans-polar of *spanning* polytope):
- **Construction** (trans-polar)
  - Decompose $\Delta^\star$ into convex faces θ_i;
  - Find the (standard) polar $(\theta_i)^\circ$ for each (convex) face
  - (Re)assemble parts dually to $(\theta_i \cap \theta_j)^\circ = [(\theta_i)^\circ, (\theta_j)^\circ]$ with "neighbors"

poset (Σ,≺)

- "Normal fan"
  - "outer" [GE]
  - "inner/*local*" [CLS]
- Dual cones ↦ inside opening vertex-cones [?BH]



- *Agrees with* ***standard*** *(if obscure?)* ***constructions***…

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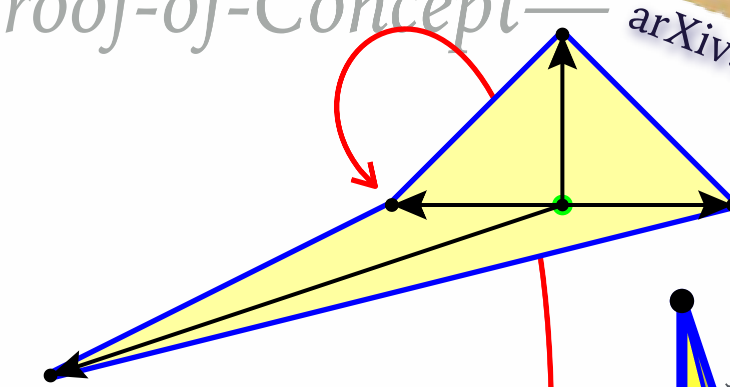
& Non-Convex Mirrors

—Proof-of-Concept— arXiv:1611.10300

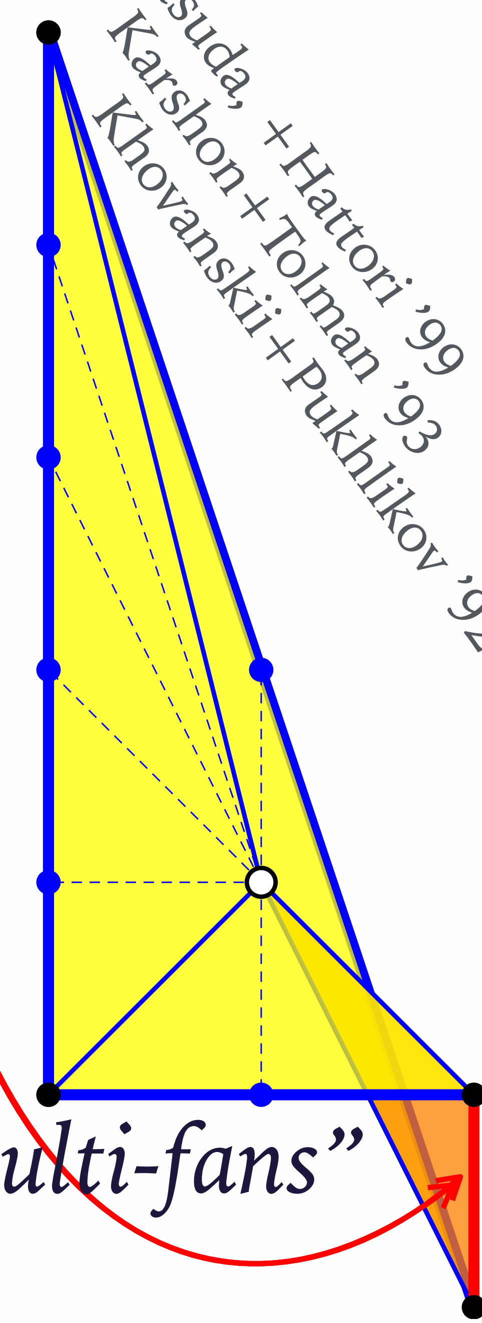


- The *oriented Newton* polytope:
  - is star-triangulable → a toric *space*
  - differs from its convex hull by “flip-folded” simplices
- Associating coordinates to corners:
  - *SP*: $x_1 = (-1,0)$, $x_2 = (1,0)$, $x_3 = (0,1)$, $x_4 = (-3,-1)$
  - *NP*: $y_1 = (-1,4)$, $y_2 = (-1,-1)$, $y_3 = (1,-1)$, $y_4 = (1,-2)$
  - Expressing each as a monomial in the others:

$$NP:\ x_1^2 x_3^5 \oplus x_1^2 x_4^5 \oplus \frac{x_2^2}{x_4} \oplus \frac{x_2^2}{x_3} \quad vs. \quad SP:\ y_1^2 y_2^2 \oplus y_3^2 y_4^2 \oplus \frac{y_1^5}{y_4} \oplus \frac{y_2^5}{y_3}$$

$$\mathbb{P}^2_{(1:1:3)}[5] \quad \begin{bmatrix} 2 & 0 & 5 & 0 \\ 2 & 0 & 0 & 5 \\ 0 & 2 & 0 & -1 \\ 0 & 2 & -1 & 0 \end{bmatrix} \overset{\text{BBHK}}{\Longleftrightarrow} \begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 5 & 0 & 0 & -1 \\ 0 & 5 & -1 & 0 \end{bmatrix} \quad \mathbb{P}^2_{(3:2:5)}[10]$$

Masuda, +Hattori ’99
Karshon+Tolman ’93
Khovanskii+Pukhlikov ’92



☛ *“multi-fans”*

Laurent GLSM Fugue

& Non-Convex Mirrors

—*Proof-of-Concept*— arXiv:1611.10300

- K3 in Hirzebruch 3-folds, "cornerstone" mirrors:

$$a_1\, x_4^8 + a_2\, x_3^8 + a_3\, \frac{x_1^3}{x_3} + a_5\, \frac{x_2^3}{x_3} : \quad \exp\left\{ 2i\pi \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{24} & \frac{1}{24} & \frac{1}{8} & 0 \\ \hline \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & \frac{1}{8} \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : \quad \begin{cases} G = \mathbb{Z}_3 \times \mathbb{Z}_{24}, \\ Q = \mathbb{Z}_8. \end{cases}$$

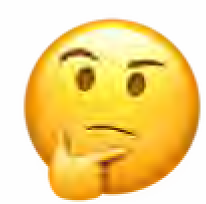
$$\begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 8 & 0 \\ 3 & 0 & -1 & 0 \\ 0 & 3 & -1 & 0 \end{bmatrix} \qquad \mathbb{P}^3_{(3:3:1:1)}[8] \qquad\qquad \mathbb{P}^3_{(3:5:8:8)}[24]/\mathbb{Z}_3$$

BBHK

$$b_1\, y_3^3 + b_2\, y_5^3 + b_3\, \frac{y_2^8}{y_3\, y_5} + b_4\, y_1^8 : \quad \exp\left\{ 2i\pi \begin{bmatrix} \frac{1}{8} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ \hline \frac{3}{24} & \frac{5}{24} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \right\} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_5 \end{bmatrix} : \quad \begin{cases} G^\nabla = \mathbb{Z}_8 \\ Q^\nabla = \mathbb{Z}_{24} \times \mathbb{Z}_3 \end{cases}$$

$$\frac{|G|}{|Q|} = \frac{3\cdot 24}{8} = 9 = \frac{d(\Delta_{\mathcal{F}_3})}{d(\Delta^\star_{\mathcal{F}_3})} = \frac{54}{6}$$

- The Hilbert space & interactions restricted by the symmetries
  - Analysis: classical, semi-classical, quantum corrections…
  - …in spite of the manifest singularity in the (super)potential

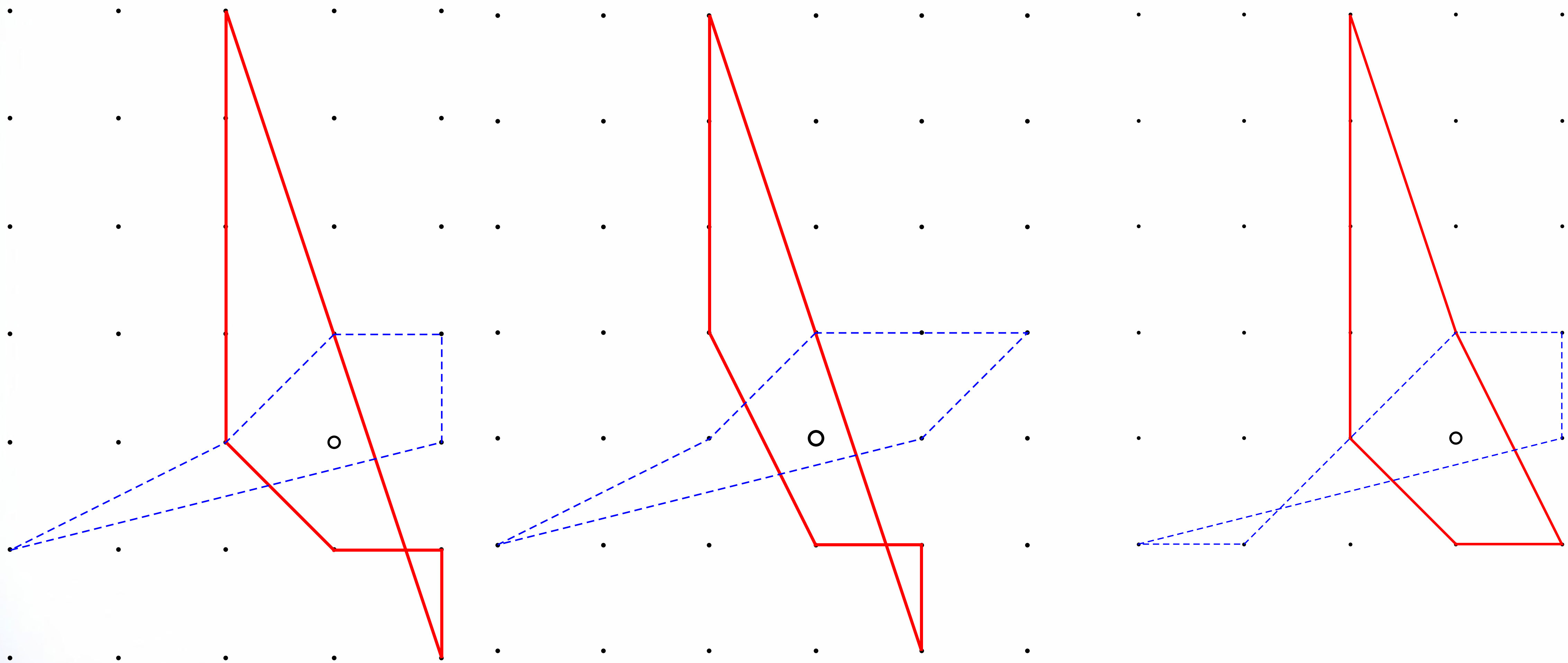


Laurent GLSM Fugue

& Non-Convex Mirrors

—*Proof-of-Concept*— arXiv:18xx.sooon

- Not just Hirzebruch *n*-folds, either:
  - Buckets of 2-dimensional polygons, invented to test $\nabla : \Delta^\star \to \Delta^\nabla$



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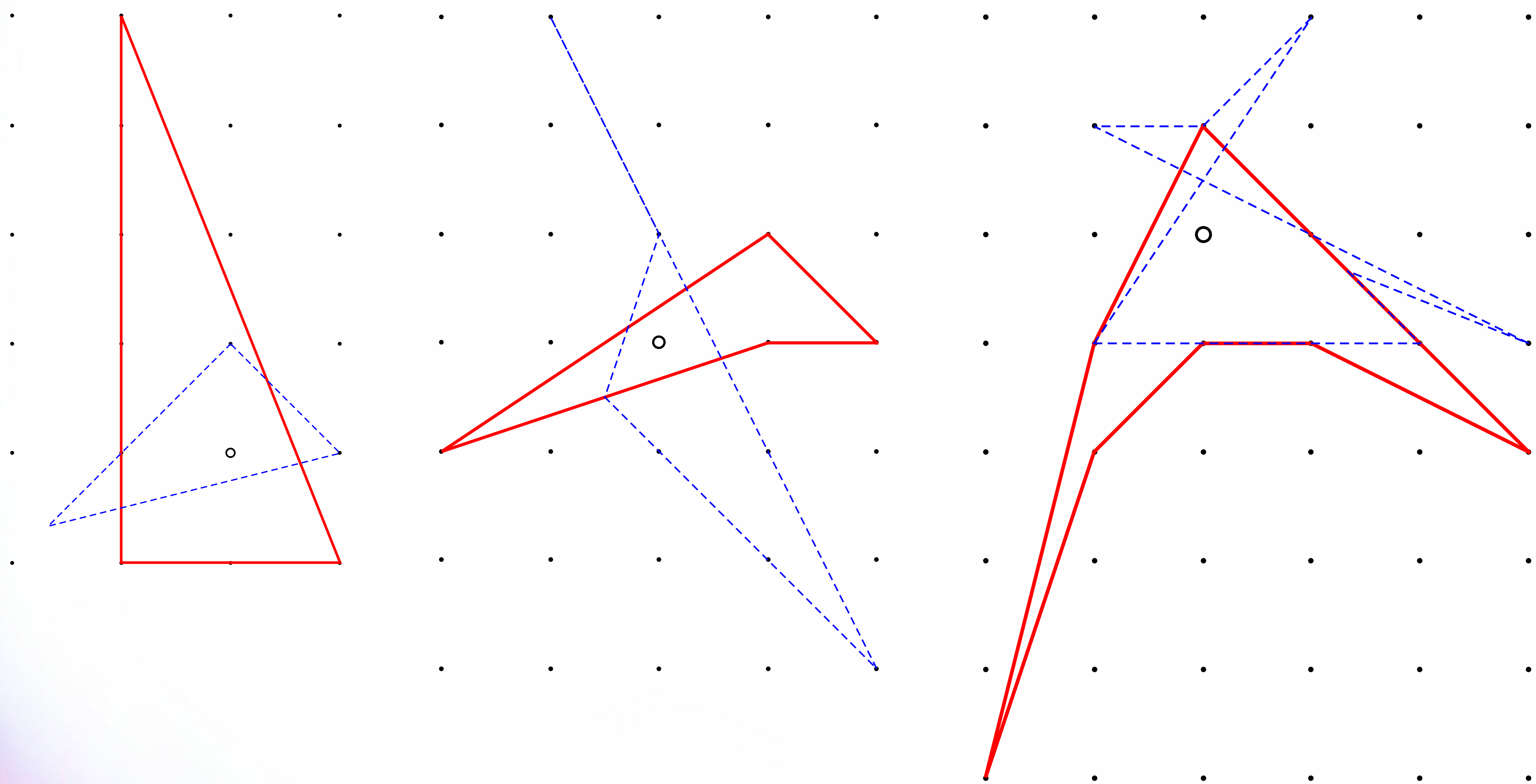

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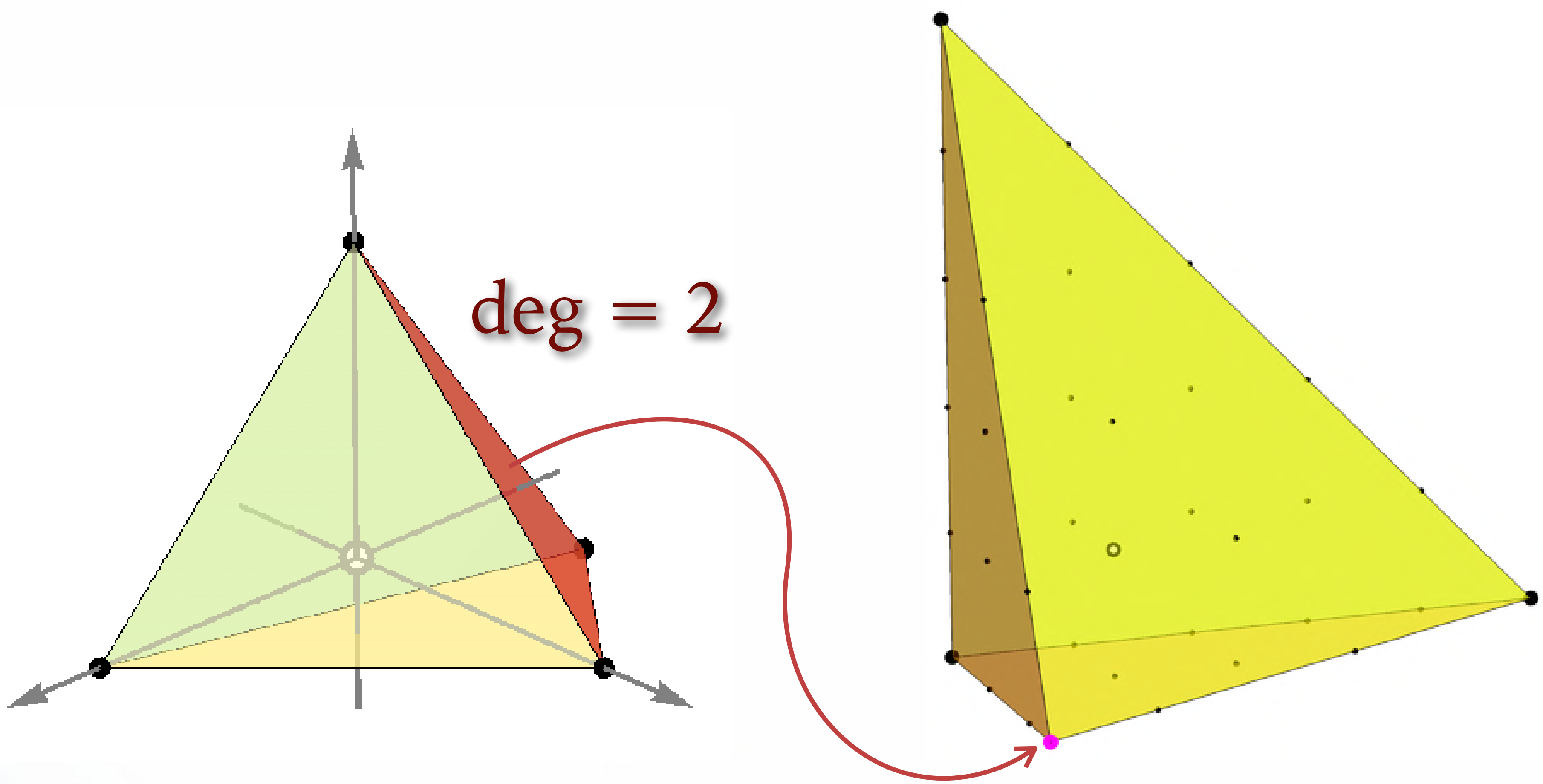
  - And, plenty of 3-dimensional polyhedra:



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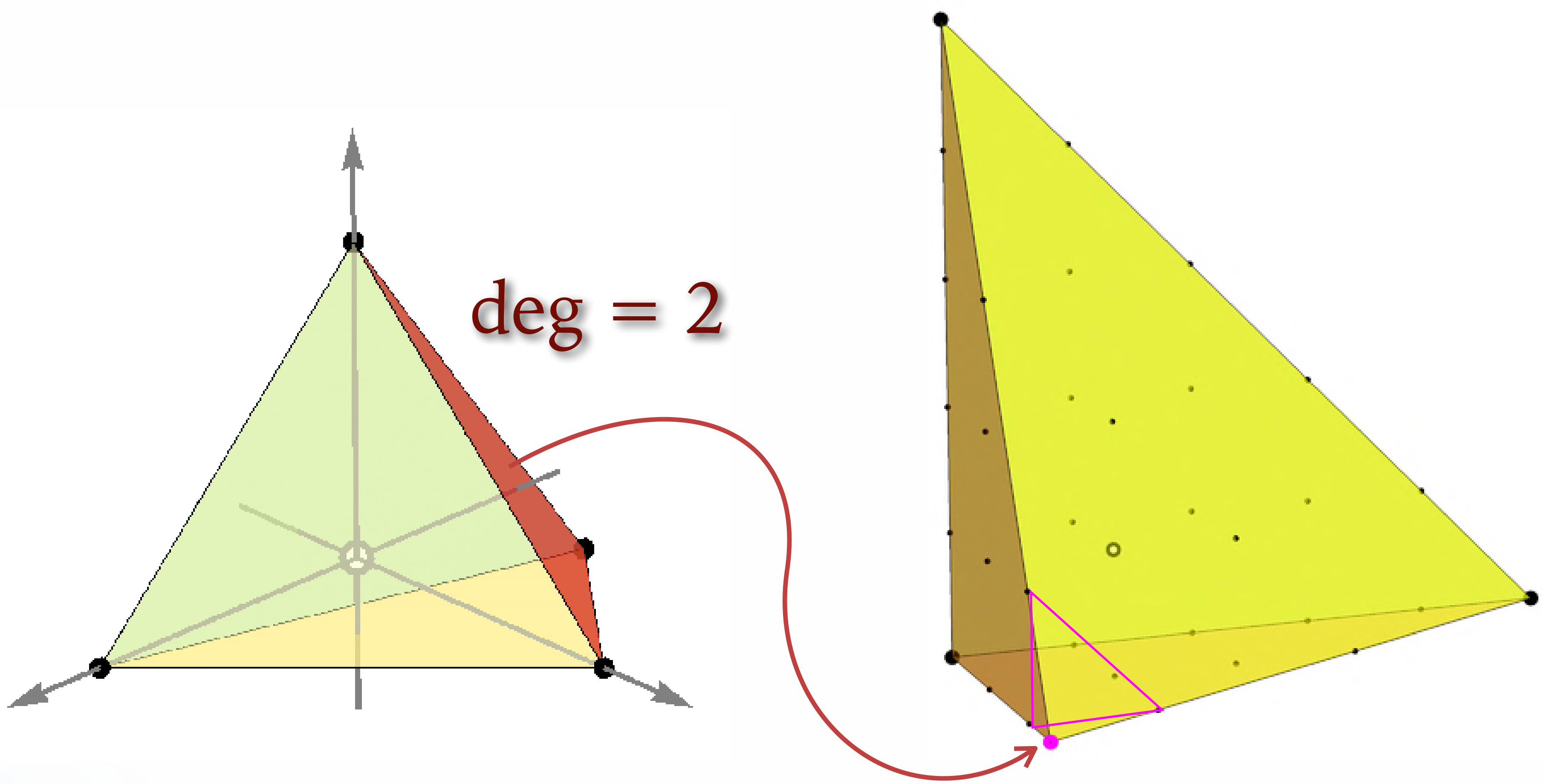

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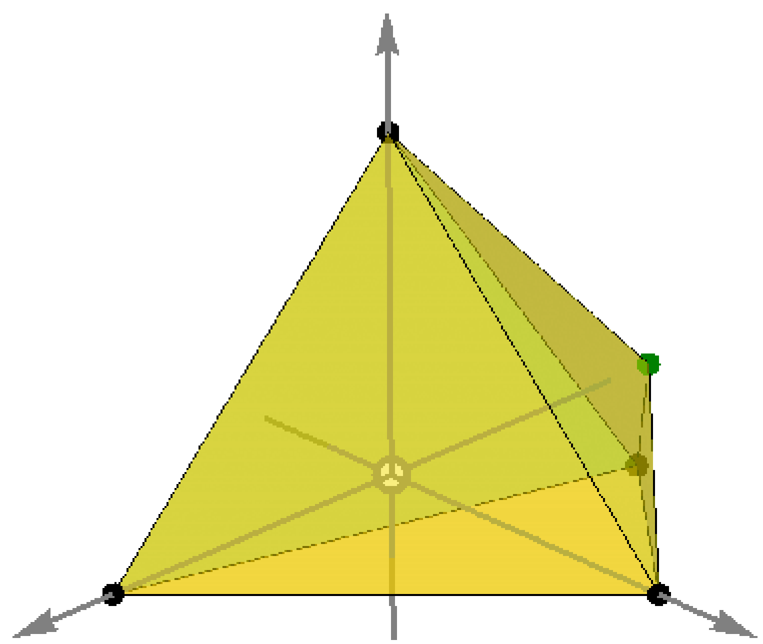

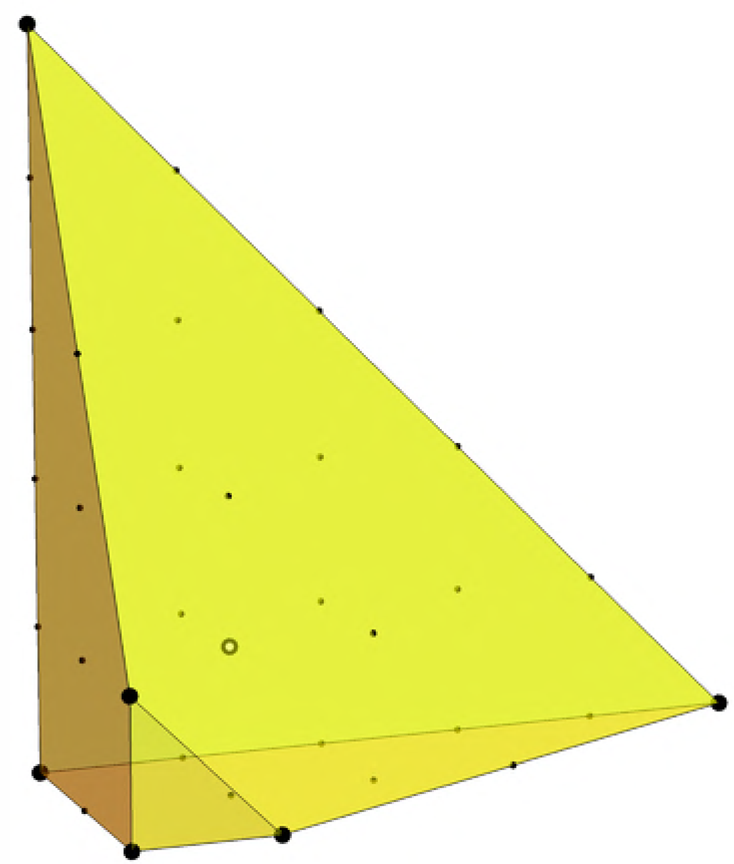

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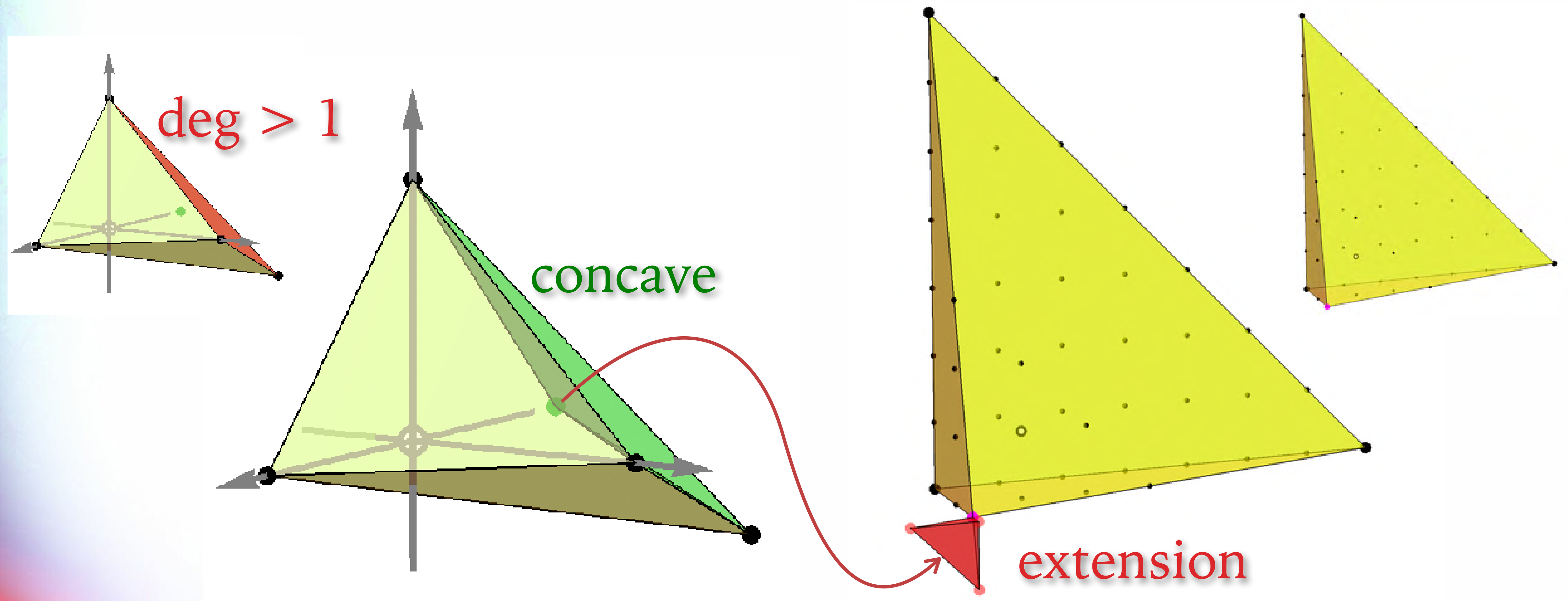

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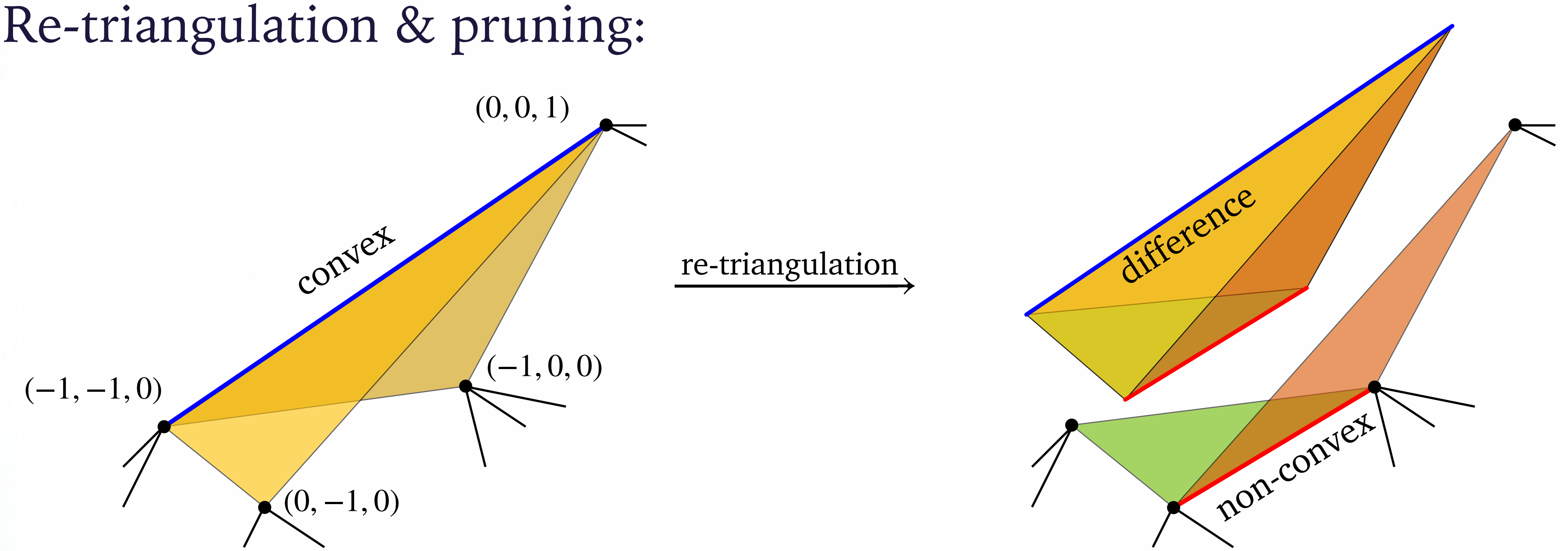
  - And, plenty of 3-dimensional polyhedra:
  - Re-triangulation & pruning:



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& Non-Convex Mirrors

—*Proof-of-Concept*—

arXiv:18xx.sooon

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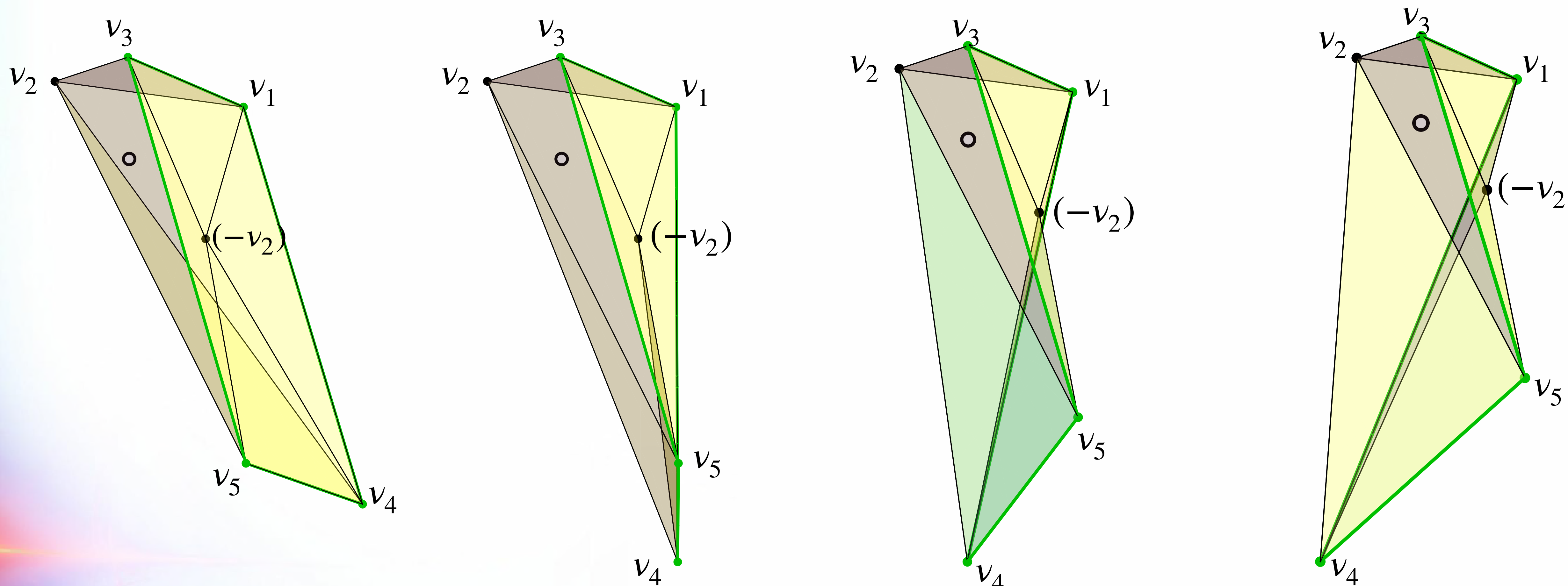
  - Buckets of 2-dimensional polygons, invented to test $^{\nabla}{:}\,\Delta^{\star} \to \Delta^{\nabla}$
  - And, plenty of 3-dimensional polyhedra:
  - Re-triangulation & pruning:
  - Multiply infinite sequences of twisted polytopes:



Laurent GLSM Fugue

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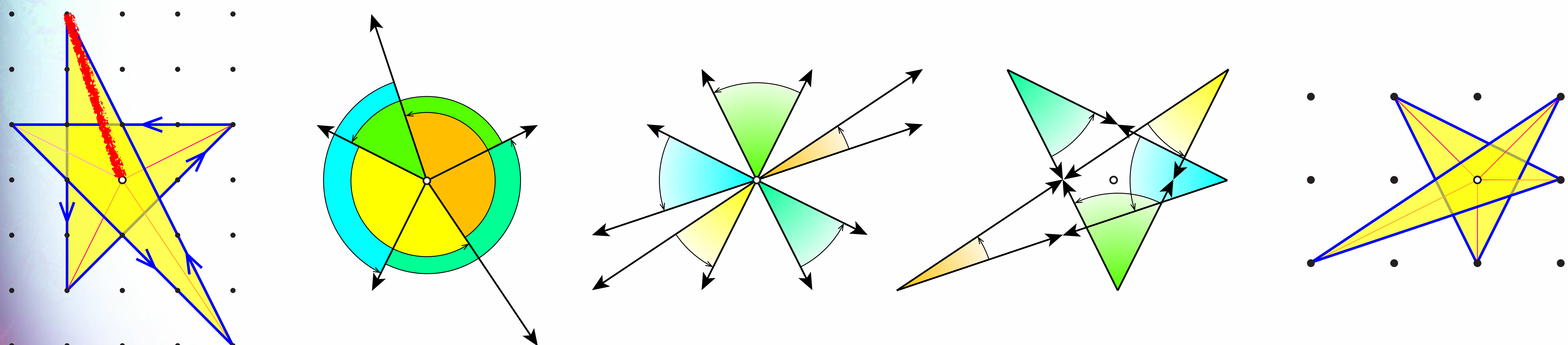
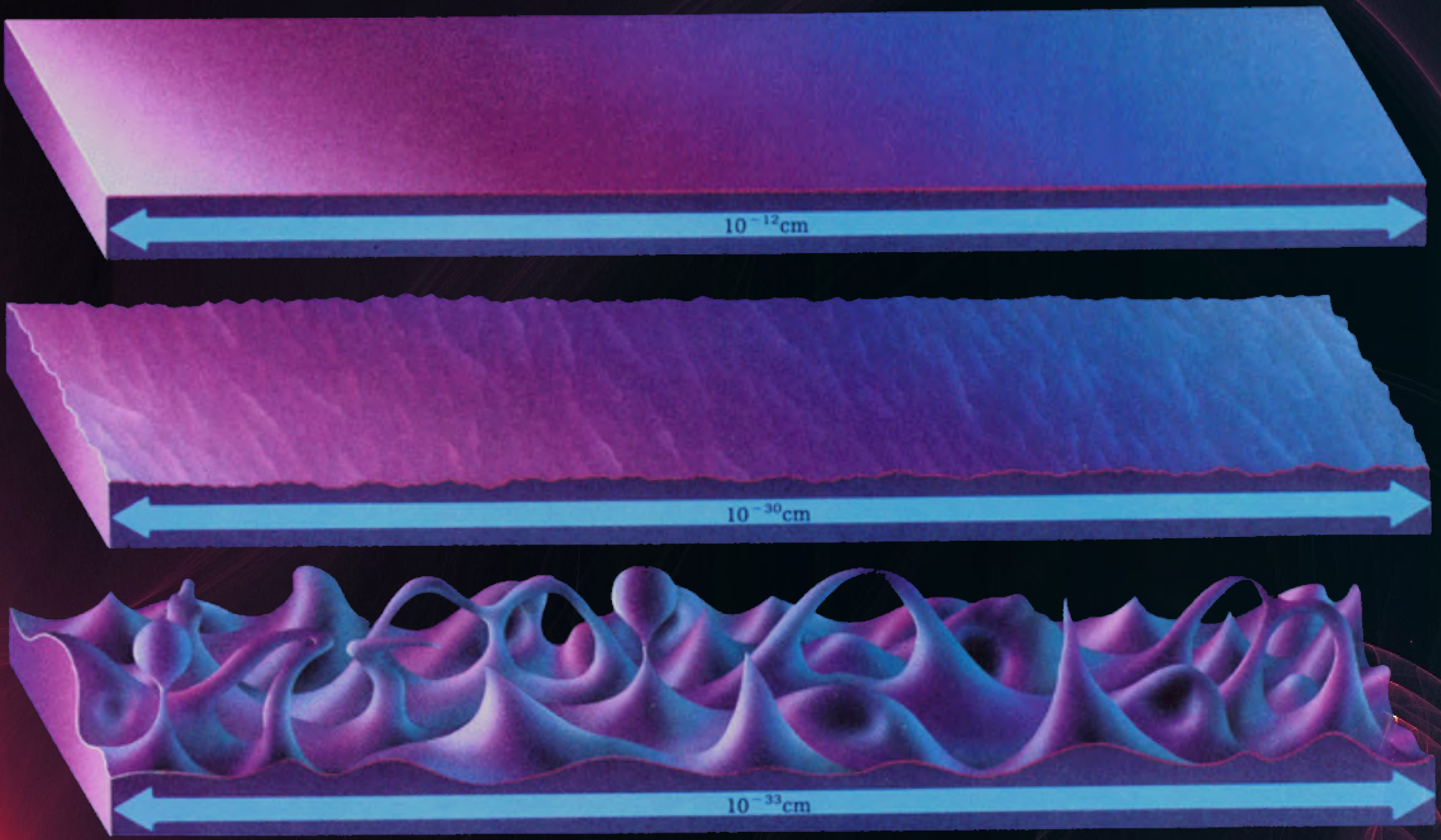
—Proof-of-Concept— arXiv:18xx.sooon

- Not just Hirzebruch *n*-folds, either:
  - Buckets of 2-dimensional polygons, invented to test $\nabla : \Delta^\star \to \Delta^\nabla$
  - And, plenty of 3-dimensional polyhedra:
  - Re-triangulation & pruning:
  - Multiply infinite sequences of twisted polytopes:
  - And multi-fans (spanned by multi-topes):



winding number (multiplicity, Duistermaat-Heckman fn.) = 2

[A. Hattori+M. Masuda" *Theory of Multi-Fans*, Osaka J. Math. **40** (2003) 1–68]



Discriminant Divertimento
(How Small Can We Go?)

Discriminant Divertimento

—*Proof-of-Concept*— arXiv:1611.10300

The Phase-Space

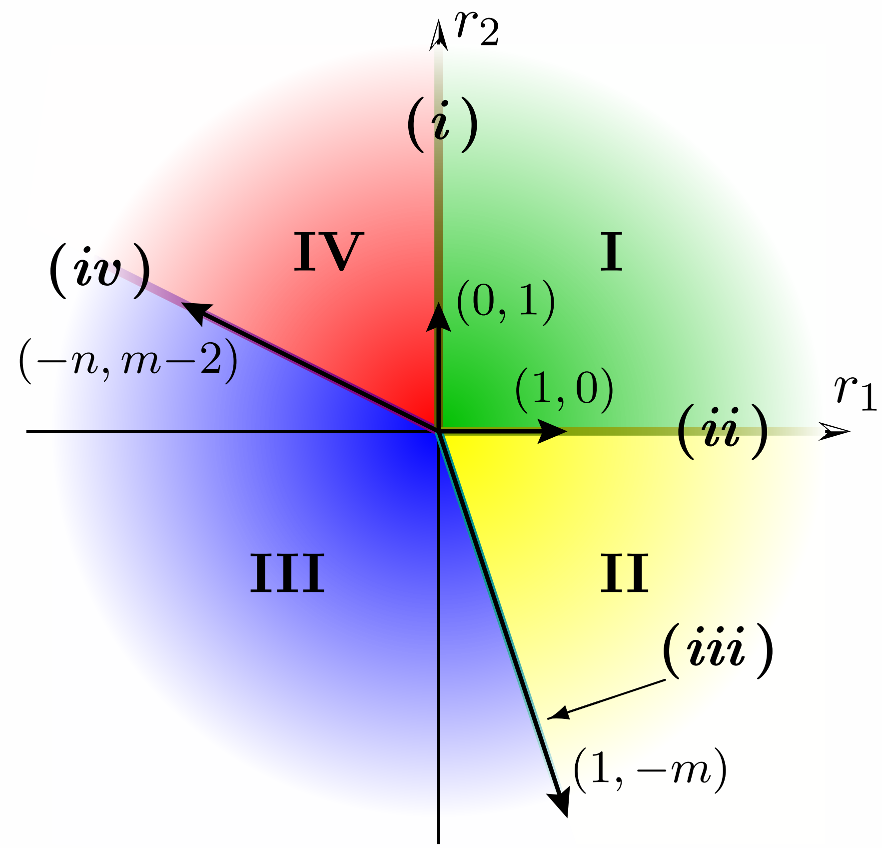
- The (super)potential: $W(X) := X_0 \cdot f(X)$,

$$f(X) := \sum_{j=1}^{2} \Big(\sum_{i=2}^{n} (a_{ij}\, X_i^n) X_{n+j}^{2-m} + a_j\, X_1^n X_{n+j}^{(n-1)m+2} \Big)$$

	X_0	X_1	X_2	$\cdots$	X_n	X_{n+1}	X_{n+2}
Q^1	$-n$	1	1	$\cdots$	1	0	0
Q^2	$m-2$	$-m$	0	$\cdots$	0	1	1

- The possible vevs

	$\lvert x_0\rvert$	$\lvert x_1\rvert$	$\lvert x_2\rvert$	$\cdots$	$\lvert x_n\rvert$	$\lvert x_{n+1}\rvert$	$\lvert x_{n+2}\rvert$
i	0	0	0	$\cdots$	0	$*$	$*$
I	0	$*$	$*$	$\cdots$	$*$	$*$	$*$
ii	0	0	$*$	$\cdots$	$*$	0	0
II	0	$\lvert x_1\rvert = \sqrt{\frac{\sum_j \lvert x_{n+j}\rvert^2 - r_2}{m}} = \sqrt{r_1 - \sum_{i=2}^{n} \lvert x_i\rvert^2} > 0$	$*$	$\cdots$	$*$	$*$	$*$
iii	0	$\sqrt{r_1}$	0	$\cdots$	0	0	0
III	$\sqrt{\frac{mr_1+r_2}{(n-1)m+2}}$	$\sqrt{\frac{(m-2)r_1+nr_2}{(n-1)m+2}}$	0	$\cdots$	0	0	0
iv	$\sqrt{-r_1/n}$	0	0	$\cdots$	0	0	0
IV	$\sqrt{-r_1/n}$	0	0	$\cdots$	0	$*$	$*$

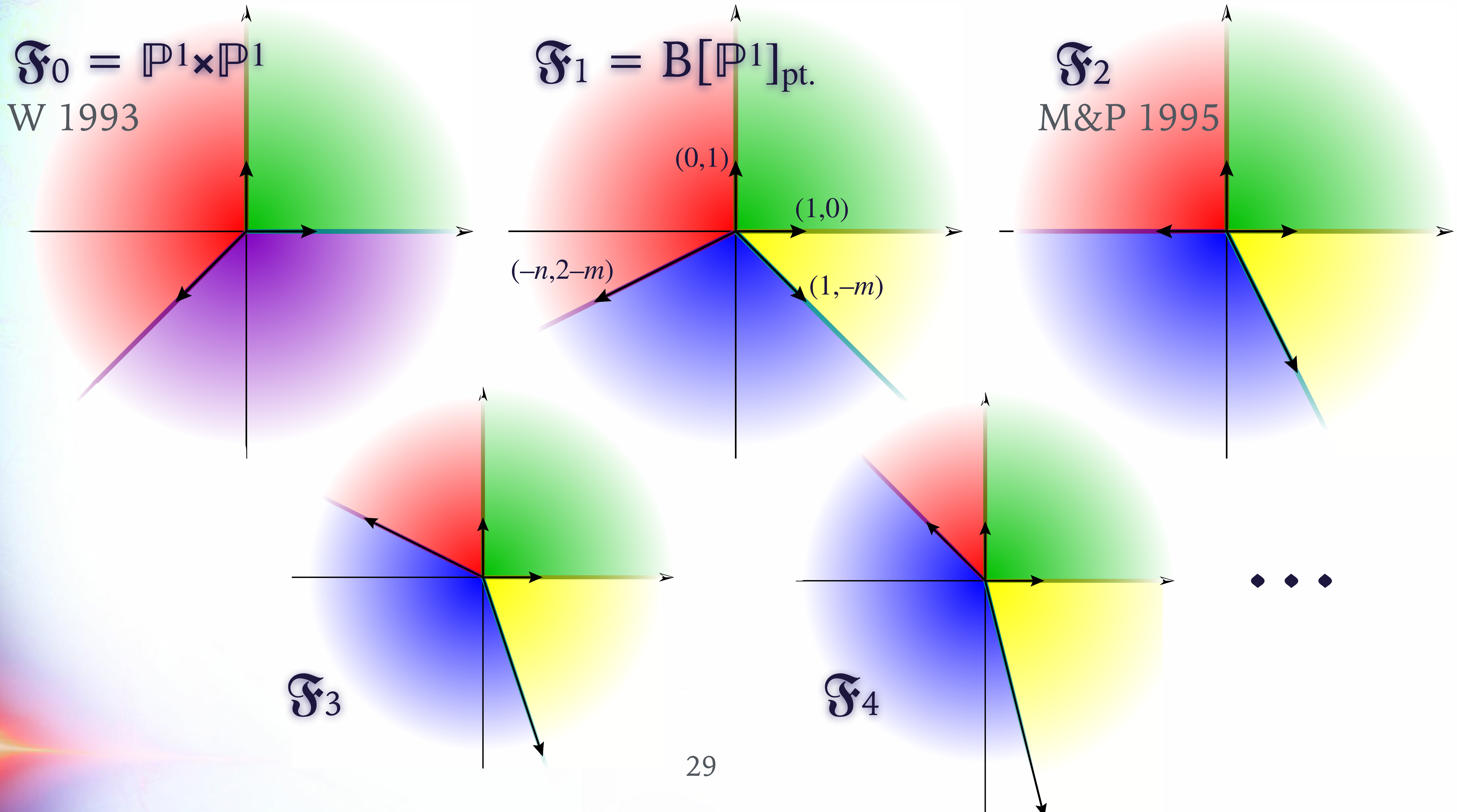


Discriminant Divertimento

—*Proof-of-Concept*— arXiv:1611.10300

The Phase-Space

- Varying m in $\mathfrak{F}_m$:



Discriminant Divertimento

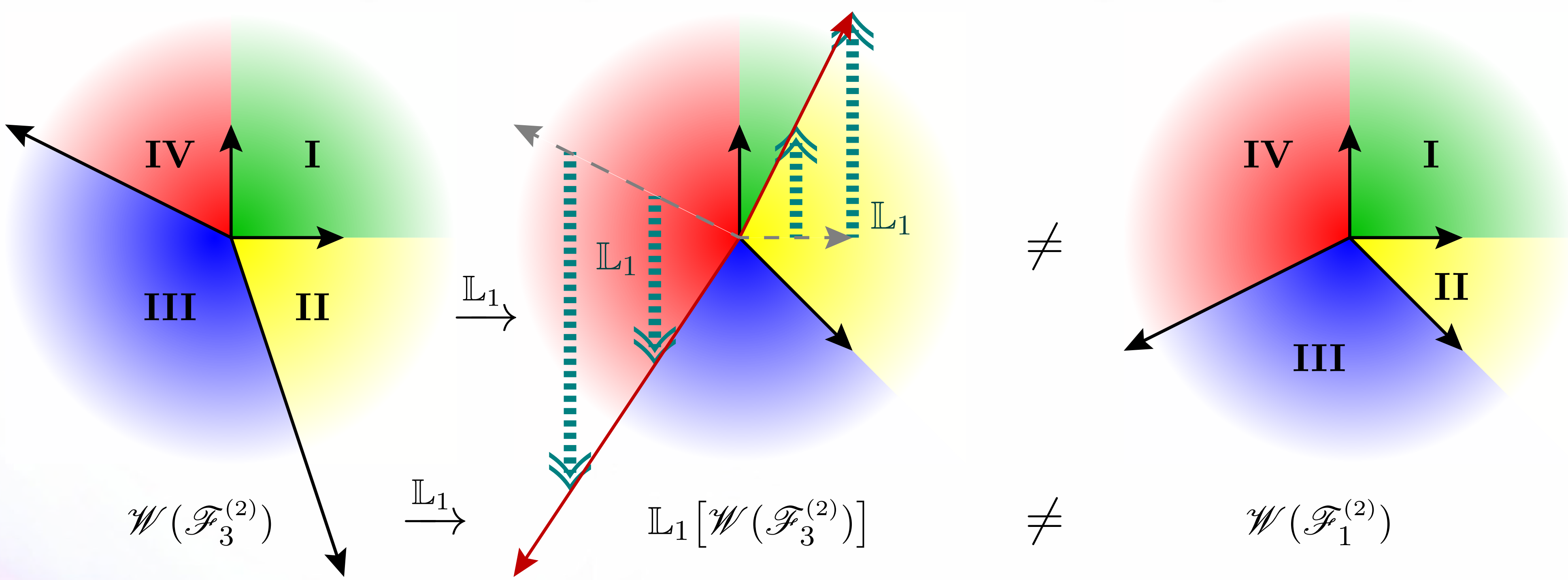
—*Proof-of-Concept*— arXiv:1611.10300

The Phase-Space

- Infinite diversity in the $\mathfrak{F}_m$:
- The $[m \pmod{n}]$ diffeomorphism $\qquad \mathbb{L}_k : \mathscr{F}_m^{(n)}[c_1] \to \mathscr{F}_{m+nk}^{(n)}[c_1]$

$$\mathbb{L}_1 : \Big\{ \overbrace{(0,1),\ (1,-m)}^{\mathscr{W}(\mathscr{F}_m^{(n)}[c_1])} \Big\} \xrightarrow{\cdot\left[\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right]} \Big\{ \overbrace{(0,1),\ (1,-(m-n))}^{\mathscr{W}(\mathscr{F}_{m-n}^{(n)}[c_1])} \Big\}$$

$$\mathbb{L}_1 : \Big\{ \overbrace{(1,0),\ (-n,m-2)}^{\mathscr{W}(\mathscr{F}_m^{(n)}[c_1])} \Big\} \xrightarrow{\cdot\left[\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right]} \{(1,n),\ (-n,m-2-n^2)\} \neq \Big\{ \overbrace{(1,0),\ (-n,(m-n)-2)}^{\mathscr{W}(\mathscr{F}_{m-n}^{(n)}[c_1])} \Big\}$$



Discriminant Divertimento

The Discriminant

—Proof-of-Concept— arXiv:RealSoon

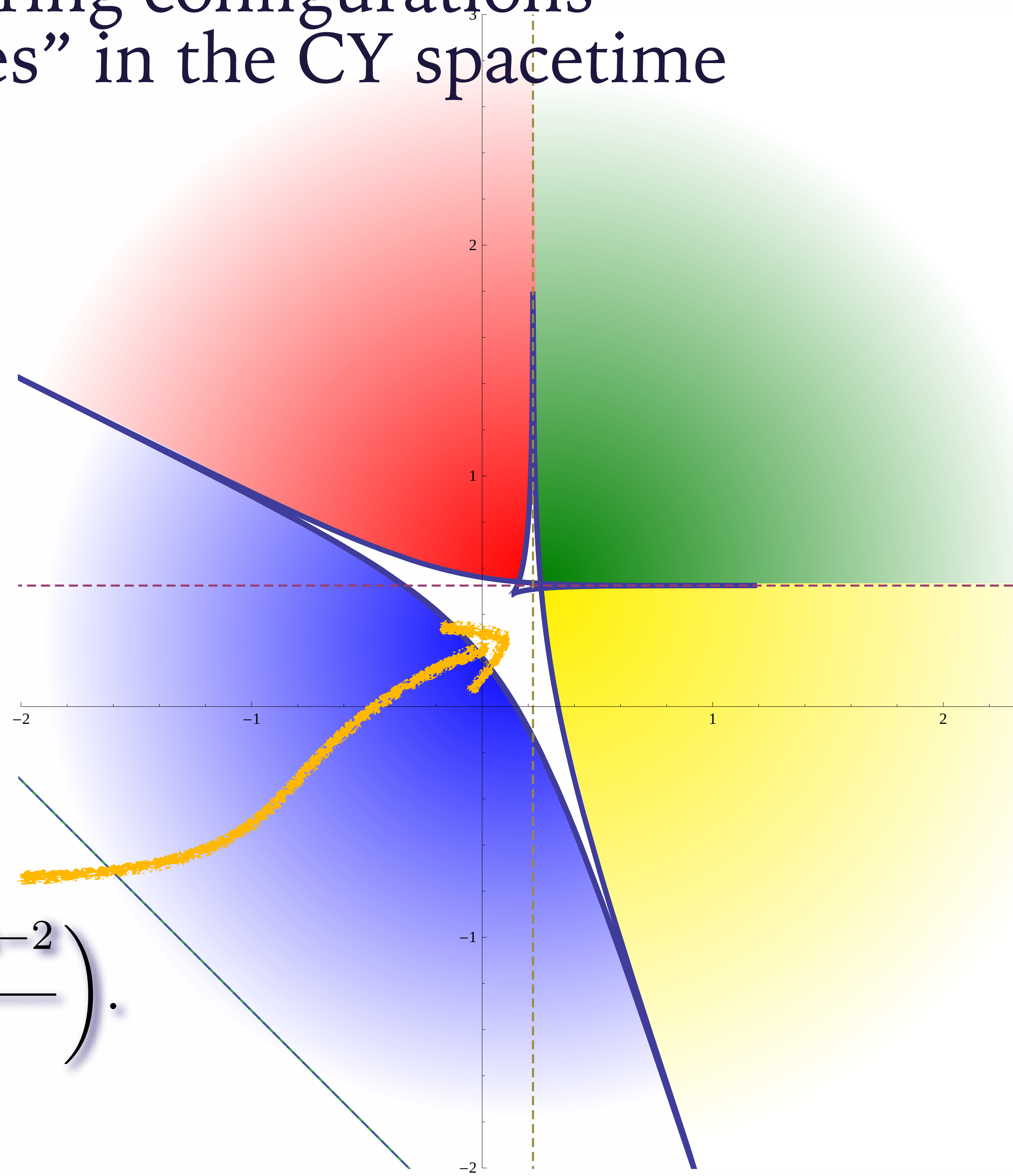
- Now add “instantons”: 0-energy string configurations wrapped around “tunnels” & “holes” in the CY spacetime
  - Near $(r_1,r_2) \sim (0,0)$, classical analysis of the Kähler (metric) phase-space fails [M&P: arXiv:hep-th/9412236]
- With

	X_0	X_1	X_2	$\cdots$	X_n	X_{n+1}	X_{n+2}
Q^1	$-n$	1	1	$\cdots$	1	0	0
Q^2	$m-2$	$-m$	0	$\cdots$	0	1	1

- the instanton resummation gives:

$$r_1 + \frac{\hat\theta_1}{2\pi i} = -\frac{1}{2\pi}\log\left(\frac{\sigma_1^{n-1}(\sigma_1 - m\,\sigma_2)}{[(m-2)\sigma_2 - n\sigma_1]^n}\right),$$

$$r_2 + \frac{\hat\theta_2}{2\pi i} = -\frac{1}{2\pi}\log\left(\frac{\sigma_2^2\,[(m-2)\sigma_2 - n\sigma_1]^{m-2}}{(\sigma_1 - m\,\sigma_2)^m}\right).$$



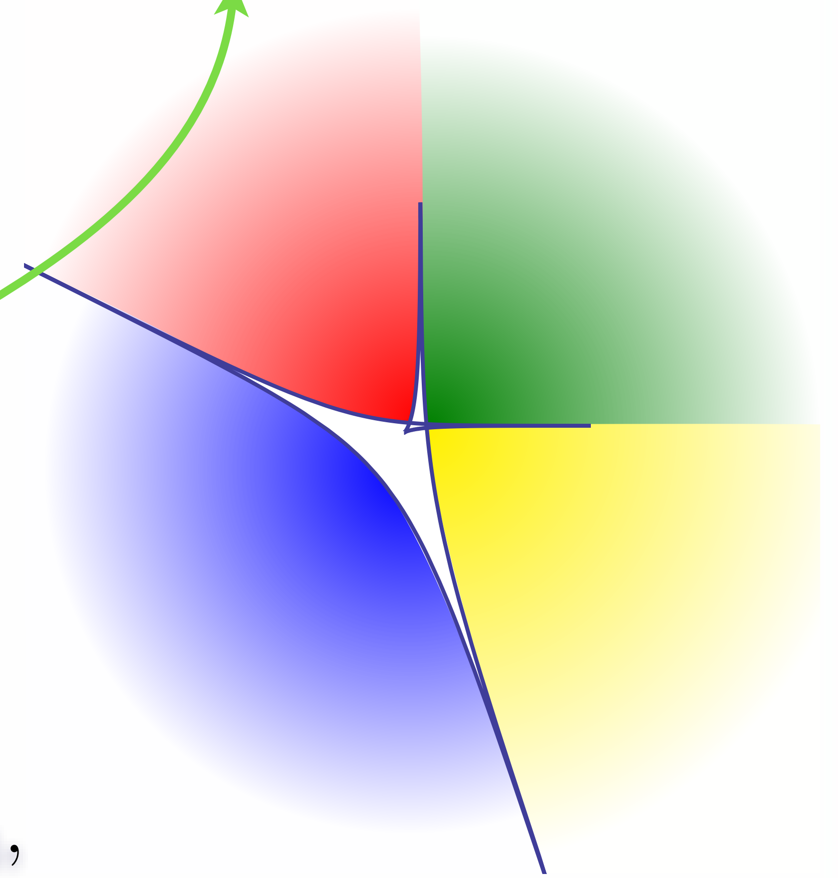
…and a Mirror Motet
(Yes, the BBHK-mirrors)

Mirror Motets

The Discriminant

—*Proof-of-Concept*— arXiv:RealSoon

- Now compare with the complex structure of the BBHK-mirror
  - Restricted to the "cornerstone" def. poly



$$f(x) = a_0 \prod_{\nu_i \in \Delta^\star} (x_{\nu_i})^{\langle \nu_i, \mu_0 \rangle + 1} + \sum_{\mu_I \in \Delta} a_{\mu_I} \prod_{\nu_i \in \Delta^\star} (x_{\nu_i})^{\langle \nu_i, \mu_I \rangle + 1}$$

Batyrev

$$g(y) = b_0 \prod_{\mu_I \in \Delta} (y_{\mu_I})^{\langle \mu_I, \nu_0 \rangle + 1} + \sum_{\nu_i \in \Delta^\star} b_{\nu_i} \prod_{\mu_I \in \Delta} (y_{\mu_I})^{\langle \mu_I, \nu_i \rangle + 1}$$

- In particular,

$$g(y) = \sum_{i=0}^{n+2} b_i\, \phi_i(y) \; = b_0\, \phi_0 + b_1\, \phi_1 + b_2\, \phi_2 + b_3\, \phi_3 + b_4\, \phi_4,$$

$$\phi_0 := y_1 \cdots y_4, \quad \phi_1 := y_1^2\, y_2^2, \quad \phi_2 := y_3^2\, y_4^2, \quad \phi_3 := \frac{y_1^{m+2}}{y_3^{m-2}}, \quad \phi_4 := \frac{y_2^{m+2}}{y_4^{m-2}},$$

$$z_1 = -\frac{\beta\,[(m-2)\beta + m]}{m+2}, \qquad z_2 = \frac{(2\beta+1)^2}{(m+2)^2\, \beta^m}, \qquad \beta := \left[\frac{b_1\, \phi_1}{b_0\, \phi_0} \Big/ {}^{A}\mathscr{J}(g)\right],$$

Mirror Motets

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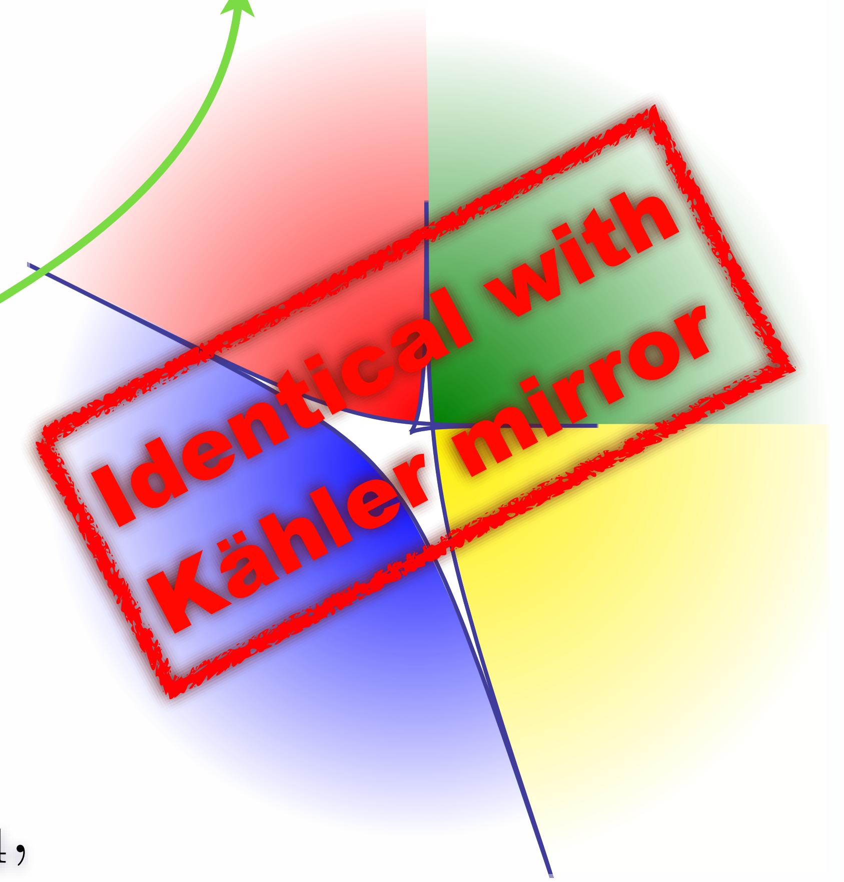
Batyrev

$$g(y) = b_0 \prod_{\mu_I\in\Delta}(y_{\mu_I})^{\langle\mu_I,\nu_0\rangle+1} + \sum_{\nu_i\in\Delta^\star} b_{\nu_i}\prod_{\mu_I\in\Delta}(y_{\mu_I})^{\langle\mu_I,\nu_i\rangle+1}$$



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Mirror Motets

The Discriminant

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- So,

$$\mathscr{W}(\mathscr{F}^{(n)}_m):\quad \begin{cases} e^{-2\pi r_1 + i\hat{\theta}_1} = \dfrac{1 - m\,\rho}{[(m{-}2)\rho - n]^n}, \\[2ex] e^{-2\pi r_2 + i\hat{\theta}_2} = \dfrac{\rho^2\,[(m{-}2)\rho - n]^{m-2}}{(1 - m\,\rho)^m}; \end{cases} \qquad \rho := \frac{\sigma_2}{\sigma_1}$$

- and

$$\mathscr{M}({}^\nabla\mathscr{F}^{(n)}_m[c_1]):\quad \begin{cases} z_1 = (-1)^{n-1}\dfrac{\beta\,[(m{-}2)\,\beta + m]^{n-1}}{[(n{-}1)m{+}2]^{n-1}}, \\[2ex] z_2 = \dfrac{(1 + n\,\beta)^2}{[(n{-}1)m{+}2]^2\,\beta^m}, \end{cases} \qquad \beta := \left[\frac{b_1\,\phi_1}{b_0\,\phi_0}\Big/{}^A\mathscr{J}(g)\right]$$

$$\gamma := \left[\frac{b_3\,\phi_3}{b_2\,\phi_2}\Big/{}^A\mathscr{J}(g)\right]$$

$$z_1 = \frac{1 - m\,\gamma}{[(m{-}2)\gamma - 2]^2}, \qquad z_2 = \frac{\gamma^2\,[(m{-}2)\gamma - 2]^{m-2}}{(1 - m\,\gamma)^m},$$

- are *identical*?!
  - You bet: $\begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} -n & m{-}2 \\ 1 & -m \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix}$

$$\mathscr{M}({}^\nabla\mathscr{F}^{(n)}_m) \supset \mathbb{P}^1 \xrightarrow{\ \approx\ } \mathbb{P}^1 \subset \mathscr{W}(\mathscr{F}^{(n)}_m)$$

Horn uniformization — Morrison+Plesser, ’93

- Better yet: $(\sigma_1, \sigma_2) \overset{\text{MM}}{\approx} (b_2\phi_2, b_3\phi_3)/{}^A\mathscr{J}$

Mirror Motets

The Discriminant

—*Proof-of-Concept*— arXiv:RealSoon

- So: $\mathscr{W}\big(\mathscr{F}^{(n)}_m[c_1]\big) \overset{\text{mm}}{\approx} \mathscr{M}\big({}^{\nabla}\mathscr{F}^{(n)}_m[c_1]\big)$
- In fact, also: $\mathscr{W}\big({}^{\nabla}\mathscr{F}^{(n)}_m[c_1]\big) \overset{\text{mm}}{\approx} \mathscr{M}\big(\mathscr{F}^{(n)}_m[c_1]\big)$

✅…when restricted to no (MPCP) blow-ups & “cornerstone” polynomial

- Then, $\dim \mathscr{W}\big({}^{\nabla}\mathscr{F}^{(n)}_m[c_1]\big) = n = \dim \mathscr{M}\big(\mathscr{F}^{(n)}_m[c_1]\big)$
- Same method:

$$e^{2\pi i\,\tilde{\tau}_\alpha} = \prod_{I=0}^{2n}\Big(\sum_{\beta=1}^{2}\widetilde{Q}^{\beta}_I\,\tilde{\sigma}_\beta\Big)^{\widetilde{Q}^{\alpha}_I}$$

$$\tilde{z}_a = \prod_{I=0}^{2n}\big(a_I\,\varphi_I(x)\big)^{\widetilde{Q}^{\alpha}_I}\Big/{}^{A}\mathscr{J}$$

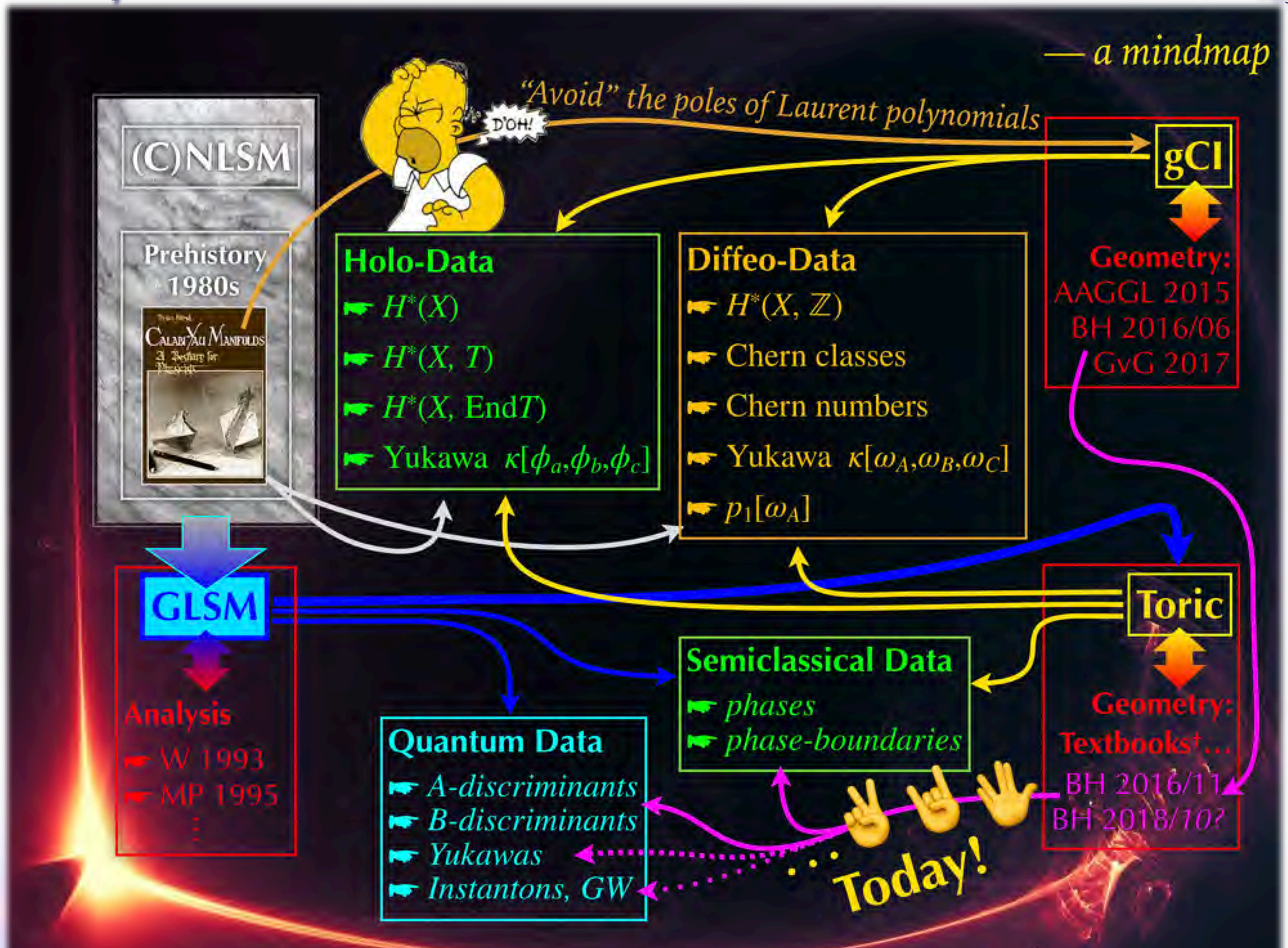
I	$\big(\sum_\beta \widetilde{Q}^{\beta}_I\,\tilde{\sigma}_\beta\big)$	$(a_I\,\varphi_I)/{}^{A}\mathscr{J}_{(210)}(f)$
0	$-2(m{+}2)(\tilde{\sigma}_1+\tilde{\sigma}_2)$	$-2\big((a_3\,\varphi_3)+(a_4\,\varphi_4)\big)$
1	$m\,\tilde{\sigma}_1+2\,\tilde{\sigma}_2$	$\frac{m\,(a_3\,\varphi_3)+2\,(a_4\,\varphi_4)}{m+2}$
2	$2\,\tilde{\sigma}_1+m\,\tilde{\sigma}_2$	$\frac{2\,(a_3\,\varphi_3)+m\,(a_4\,\varphi_4)}{m+2}$
3	$(m{+}2)\,\tilde{\sigma}_1$	$(a_3\,\varphi_3)$
4	$(m{+}2)\,\tilde{\sigma}_2$	$(a_4\,\varphi_4)$

Laurent GLSM Coda

Summary

—Proof-of-Concept—

arXiv:1611.10300 + more



Laurent GLSM Coda

Summary

—*Proof-of-Concept*— arXiv:1611.10300 + more

- CY(n–1)-folds in Hirzebruch 4-folds
  - Euler characteristic ✅
  - Chern class, term-by-term ✅
  - Hodge numbers ✅
  - Cornerstone polynomials & mirror ✅
  - Phase-space regions & mirror ✅
  - Phase-space discriminant & mirror ✅
  - The "other way around" ✅ *(limited)*
  - Yukawa couplings ✅
  - World-sheet instantons ✅
  - Gromov-Witten invariants SOON ✅
    - *Will there be anything else?*

$d(\theta^{(k)}) := k!\ \mathrm{Vol}(\theta^{(k)})$ [BH: *signed* by orientation!]

- Oriented polytopes
- Trans-polar$^\triangledown$ constr.
- Newton $\Delta_X := (\Delta^\star{}_X)^\triangledown$
- VEX polytopes
  s.t.: $((\Delta)^\triangledown)^\triangledown = \Delta$
- Star-triangulable
  w/flip-folded faces
- Polytope extension
  ⇔ Laurent monomials

BBHK mirrors

Textbooks to be (re)written, amended

Thank You!

http://physics1.howard.edu/~thubsch/

Departments of Physics & Astronomy and Mathematics, Howard University, Washington DC
Department of Physics, Faculty of Natural Sciences, Novi Sad University, Serbia
Department of Physics, University of Central Florida, Orlando FL

$$\left\{\left\{\Delta^{\star}= \begin{pmatrix} 3 & 0 & -2 & -1 & -1 & -2 & 0 & 1 \\ -2 & 1 & -1 & -1 & -2 & -5 & -1 & -1 \end{pmatrix}, \Sigma= ,\right.\right.$$

$$\left\{\begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -2 & -1 \end{pmatrix}, \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}, \begin{pmatrix} -1 & -2 \\ -2 & -5 \end{pmatrix}, \begin{pmatrix} -2 & -5 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}\right\}, |\sigma|= , \{3, 2, 1, 1, 1, 2, 1, 1\},$$

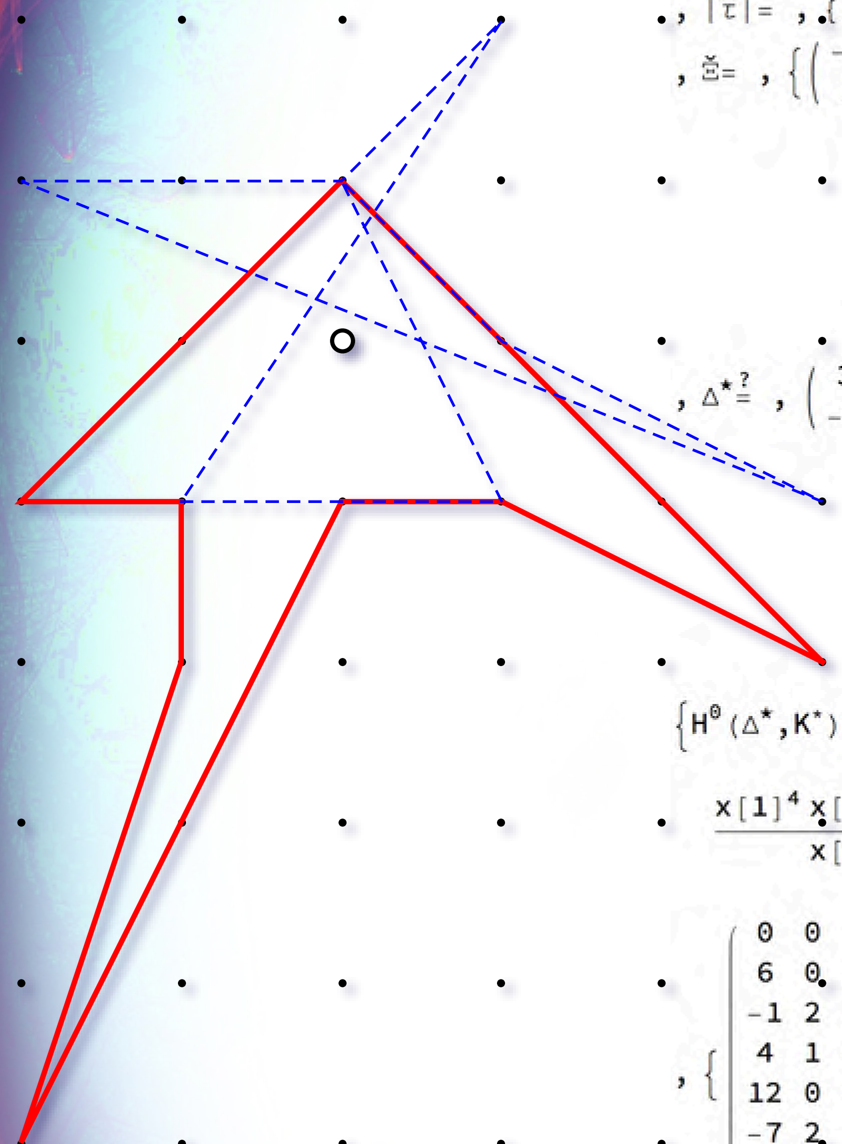
$$, \check{\Sigma}= , \left\{\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -5 & 2 \\ 2 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 5 \\ 0 & -2 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ -3 & 1 \end{pmatrix}\right\},$$

$$, \Delta := (\Delta^{\star})^{\triangledown} = , \begin{pmatrix} -1 & 1 & 0 & 1 & 3 & -2 & 0 & 1 \\ -1 & -1 & 1 & 0 & -1 & 1 & 1 & 2 \end{pmatrix}, \text{Integral},$$

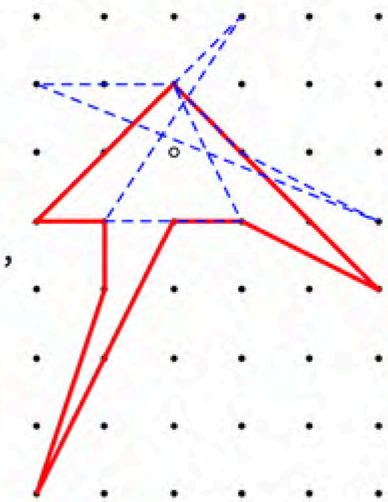
$$, \Xi= , \left\{\begin{pmatrix} 1 & 2 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}\right\},$$

$$, |\tau|= , \{1, 2, 1, -1, -1, 1, -2, -1\},$$

$$, \check{\Xi}= , \left\{\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -2 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}\right\},$$



$$, \Delta^{\star} \stackrel{?}{=} , \begin{pmatrix} 3 & 0 & -2 & -1 & -1 & -2 & 0 & 1 \\ -2 & 1 & -1 & -1 & -2 & -5 & -1 & -1 \end{pmatrix}, \text{Involution}\Big\},$$



$$\Big\}$$

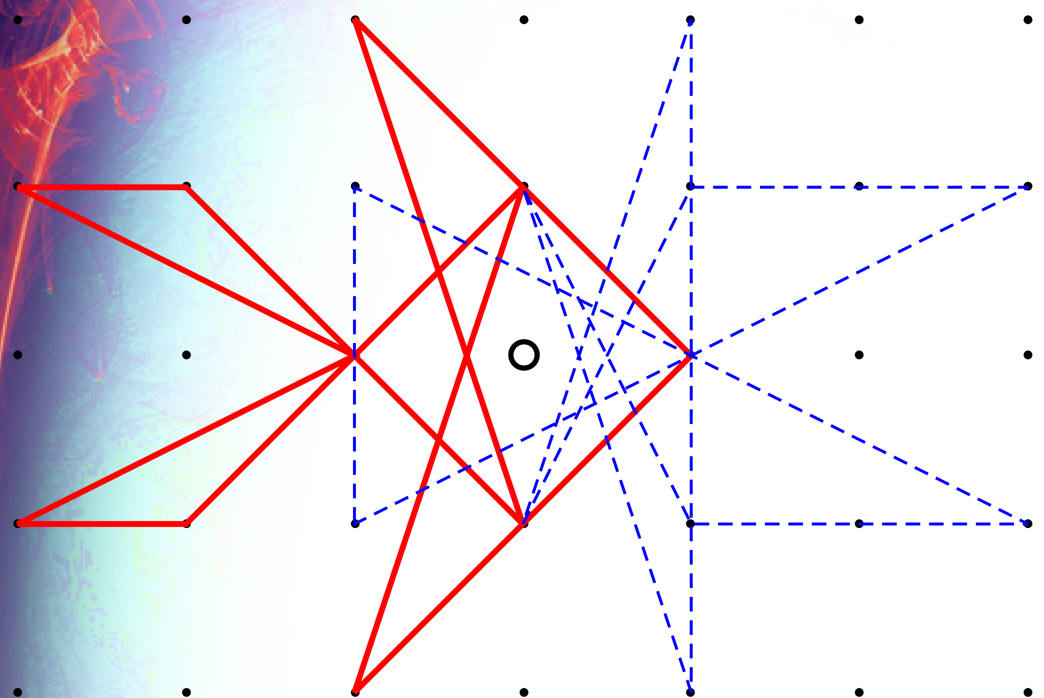
$$\Big\{H^{0}(\Delta^{\star},K^{\star}), \Big\{x[3]^4 x[4]^3 x[5]^4 x[6]^8 x[7]^2 x[8], x[1]^6 x[4] x[5]^2 x[6]^4 x[7]^2 x[8]^3, \frac{x[2]^2}{x[1]\, x[5]\, x[6]^4},$$

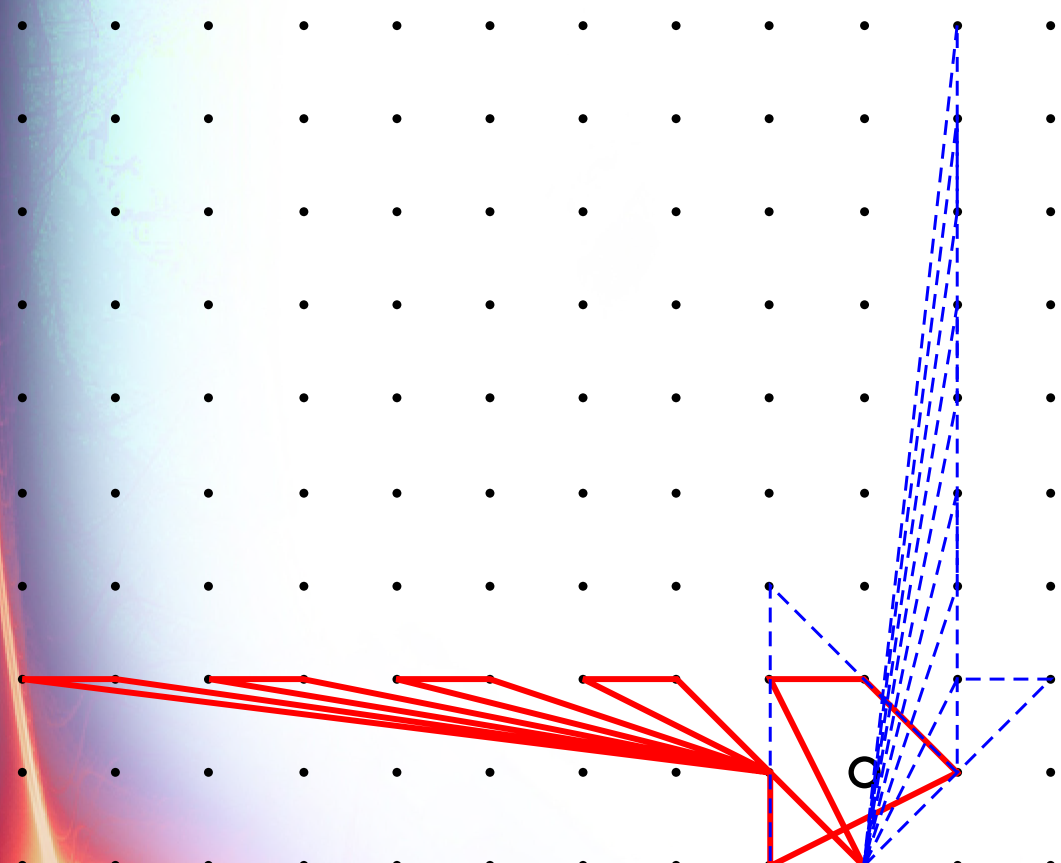
$$\frac{x[1]^4 x[2] x[7] x[8]^2}{x[3]\, x[6]}, \frac{x[1]^{12} x[7]^2 x[8]^5}{x[3]^4 x[4]}, \frac{x[2]^2 x[3]^4 x[4]^2 x[5]}{x[1]^7 x[8]^2}, \frac{x[2]^2}{x[1]\, x[5]\, x[6]^4}, \frac{x[2]^3}{x[3]^3 x[4]^2 x[5]^4 x[6]^{11} x[7]}\Big\},$$

$$, \left\{\begin{pmatrix} 0 & 0 & 4 & 3 & 4 & 8 & 2 & 1 \\ 6 & 0 & 0 & 1 & 2 & 4 & 2 & 3 \\ -1 & 2 & 0 & 0 & -1 & -4 & 0 & 0 \\ 4 & 1 & -1 & 0 & 0 & -1 & 1 & 2 \\ 12 & 0 & -4 & -1 & 0 & 0 & 2 & 5 \\ -7 & 2 & 4 & 2 & 1 & 0 & 0 & -2 \\ -1 & 2 & 0 & 0 & -1 & -4 & 0 & 0 \\ 0 & 3 & -3 & -2 & -4 & -11 & -1 & 0 \end{pmatrix}, 0\right\},$$

$$, H^{0}(\Delta^{\triangledown},K^{\star}), \Big\{\frac{y[2]^6 y[4]^4 y[5]^{12}}{y[3]\, y[6]^7 y[7]}, y[3]^2 y[4] y[6]^2 y[7]^2 y[8]^3, \frac{y[1]^4 y[6]^4}{y[4]\, y[5]^4 y[8]^3},$$

$$\frac{y[1]^3 y[2] y[6]^2}{y[5]\, y[8]^2}, \frac{y[1]^4 y[2]^2 y[6]}{y[3]\, y[7]\, y[8]^4}, \frac{y[1]^8 y[2]^4}{y[3]^4 y[4]\, y[7]^4 y[8]^{11}}, \frac{y[1]^2 y[2]^2 y[4] y[5]^2}{y[8]}, \frac{y[1]\, y[2]^3 y[4]^2 y[5]^5}{y[6]^2}\Big\},$$





$$\{\{\Delta^{\star}= \begin{pmatrix}1&-1&0&-2&-3&-1&-3&-2&0&-1\\0&2&-1&1&1&0&-1&-1&1&-2\end{pmatrix}, \Sigma= , \{\begin{pmatrix}1&0\\-1&2\end{pmatrix}, \begin{pmatrix}-1&2\\0&-1\end{pmatrix}, \begin{pmatrix}0&-1\\-2&1\end{pmatrix}, \begin{pmatrix}-2&1\\-3&1\end{pmatrix},$$

$$\begin{pmatrix}-3&1\\-1&0\end{pmatrix}, \begin{pmatrix}-1&0\\-3&-1\end{pmatrix}, \begin{pmatrix}-3&-1\\-2&-1\end{pmatrix}, \begin{pmatrix}-2&-1\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\-1&-2\end{pmatrix}, \begin{pmatrix}-1&-2\\1&0\end{pmatrix}\}, |\sigma| = , \{2, 1, -2, 1, 1, 1, 1, -2, 1, 2\},$$

$$, \check{\Sigma}= , \{\begin{pmatrix}2&0\\1&1\end{pmatrix}, \begin{pmatrix}-1&-2\\0&-1\end{pmatrix}, \begin{pmatrix}1&1\\2&0\end{pmatrix}, \begin{pmatrix}1&-1\\3&-2\end{pmatrix}, \begin{pmatrix}0&-1\\1&-3\end{pmatrix}, \begin{pmatrix}-1&0\\3&-1\end{pmatrix}, \begin{pmatrix}-1&1\\2&-3\end{pmatrix}, \begin{pmatrix}1&1\\0&-2\end{pmatrix}, \begin{pmatrix}-2&-1\\1&0\end{pmatrix}, \begin{pmatrix}0&2\\-1&-1\end{pmatrix}\},$$

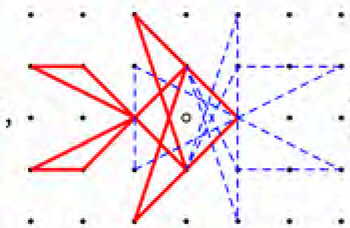
$$, \Delta:=(\Delta^{\star})^{\triangledown}= , \begin{pmatrix}-1&3&1&0&1&1&0&1&3&-1\\-1&1&1&-1&2&-2&1&-1&-1&1\end{pmatrix}, \text{Integral},$$

$$, \Xi= , \{\begin{pmatrix}-1&1\\-1&-1\end{pmatrix}, \begin{pmatrix}-1&-1\\3&1\end{pmatrix}, \begin{pmatrix}3&1\\1&1\end{pmatrix}, \begin{pmatrix}1&1\\0&-1\end{pmatrix}, \begin{pmatrix}0&-1\\1&2\end{pmatrix}, \begin{pmatrix}1&2\\1&-2\end{pmatrix}, \begin{pmatrix}1&-2\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\1&-1\end{pmatrix}, \begin{pmatrix}1&-1\\3&-1\end{pmatrix}, \begin{pmatrix}3&-1\\-1&1\end{pmatrix}\},$$

$$, |\tau| = , \{2, 2, 2, -1, 1, -4, 1, -1, 2, 2\},$$

$$, \check{\Xi}= , \{\begin{pmatrix}-1&-1\\1&-1\end{pmatrix}, \begin{pmatrix}1&1\\-3&-1\end{pmatrix}, \begin{pmatrix}1&-1\\-1&3\end{pmatrix}, \begin{pmatrix}-1&-1\\0&1\end{pmatrix}, \begin{pmatrix}2&1\\-1&0\end{pmatrix}, \begin{pmatrix}-2&-2\\-1&1\end{pmatrix}, \begin{pmatrix}1&2\\0&1\end{pmatrix}, \begin{pmatrix}-1&-1\\-1&0\end{pmatrix}, \begin{pmatrix}-1&1\\-3&1\end{pmatrix}, \begin{pmatrix}1&1\\1&3\end{pmatrix}\},$$

$$, \Delta^{\star} \overset{?}{=} , \begin{pmatrix}1&-1&0&-2&-3&-1&-3&-2&0&-1\\0&2&-1&1&1&0&-1&-1&1&-2\end{pmatrix}, \text{Involution}\},$$



$$\}$$

$$\{H^{0}(\Delta^{*},K^{*}), \{x[3]^2 x[4]^2 x[5]^3 x[6]^2 x[7]^5 x[8]^4 x[10]^4, \frac{x[1]^4 x[9]^2}{x[4]^4 x[5]^7 x[6]^2 x[7]^9 x[8]^6 x[10]^4}, \frac{x[1]^2 x[2]^2 x[9]^2}{x[5]\, x[7]^3 x[8]^2 x[10]^2},$$

$$\frac{x[1]\, x[3]^2 x[6]\, x[7]^2 x[8]^2 x[10]^3}{x[2]}, \frac{x[1]^2 x[2]^4 x[4]\, x[9]^3}{x[3]\, x[7]^4 x[8]^3 x[10]^4}, \frac{x[1]^2 x[3]^3 x[8]\, x[10]^4}{x[2]^4 x[4]^3 x[5]^4 x[9]}, \frac{x[1]\, x[2]^3 x[4]^2 x[5]^2 x[6]\, x[9]^2}{x[10]},$$

$$\frac{x[1]^2 x[3]^2 x[10]^2}{x[2]^2 x[4]^2 x[5]^3 x[7]}, \frac{x[1]^4 x[3]^2}{x[2]^4 x[4]^6 x[5]^9 x[6]^2 x[7]^7 x[8]^4}, x[2]^4 x[4]^4 x[5]^5 x[6]^2 x[7]^3 x[8]^2 x[9]^2\},$$

$$\{\{\Delta^{\star}= \begin{pmatrix}1&0&-1&0&-2&-3&-1&-4&-5&-1&-6&-7&-1&-8&-9&-1&-1\\0&1&1&-1&1&1&0&1&1&0&1&1&0&1&1&0&-1\end{pmatrix}, \Sigma= ,$$

$$\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\-1&1\end{pmatrix}, \begin{pmatrix}-1&1\\0&-1\end{pmatrix}, \begin{pmatrix}0&-1\\-2&1\end{pmatrix}, \begin{pmatrix}-2&1\\-3&1\end{pmatrix}, \begin{pmatrix}-3&1\\-1&0\end{pmatrix}, \begin{pmatrix}-1&0\\-4&1\end{pmatrix}, \begin{pmatrix}-4&1\\-5&1\end{pmatrix}, \begin{pmatrix}-5&1\\-1&0\end{pmatrix}, \begin{pmatrix}-1&0\\-6&1\end{pmatrix}, \begin{pmatrix}-6&1\\-7&1\end{pmatrix}, \begin{pmatrix}-7&1\\-1&0\end{pmatrix},$$

$$\begin{pmatrix}-1&0\\-8&1\end{pmatrix}, \begin{pmatrix}-8&1\\-9&1\end{pmatrix}, \begin{pmatrix}-9&1\\-1&0\end{pmatrix}, \begin{pmatrix}-1&0\\-1&-1\end{pmatrix}, \begin{pmatrix}-1&-1\\1&0\end{pmatrix}\}, |\sigma| = , \{1, 1, 1, -2, 1, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1, 1, 1\},$$

$$, \check{\Sigma}= , \{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}1&-1\\1&0\end{pmatrix}, \begin{pmatrix}-1&-1\\0&-1\end{pmatrix}, \begin{pmatrix}1&1\\2&0\end{pmatrix}, \begin{pmatrix}1&-1\\3&-2\end{pmatrix}, \begin{pmatrix}0&-1\\1&-3\end{pmatrix}, \begin{pmatrix}1&0\\4&-1\end{pmatrix}, \begin{pmatrix}1&-1\\5&-4\end{pmatrix}, \begin{pmatrix}0&-1\\1&-5\end{pmatrix},$$

$$\begin{pmatrix}1&0\\6&-1\end{pmatrix}, \begin{pmatrix}1&-1\\7&-6\end{pmatrix}, \begin{pmatrix}0&-1\\1&-7\end{pmatrix}, \begin{pmatrix}1&0\\8&-1\end{pmatrix}, \begin{pmatrix}1&-1\\9&-8\end{pmatrix}, \begin{pmatrix}0&-1\\1&-9\end{pmatrix}, \begin{pmatrix}-1&0\\1&-1\end{pmatrix}, \begin{pmatrix}0&1\\-1&-1\end{pmatrix}\},$$

$$, \Delta:=(\Delta^{\star})^{\triangledown}= , \begin{pmatrix}-1&0&2&1&0&1&1&0&1&1&0&1&1&0&1&1&-1\\-1&-1&1&1&-1&2&3&-1&4&5&-1&6&7&-1&8&0&2\end{pmatrix}, \text{Integral},$$

$$, \Xi= , \{\begin{pmatrix}-1&2\\-1&-1\end{pmatrix}, \begin{pmatrix}-1&-1\\0&-1\end{pmatrix}, \begin{pmatrix}0&-1\\2&1\end{pmatrix}, \begin{pmatrix}2&1\\1&1\end{pmatrix}, \begin{pmatrix}1&1\\0&-1\end{pmatrix}, \begin{pmatrix}0&-1\\1&2\end{pmatrix},$$

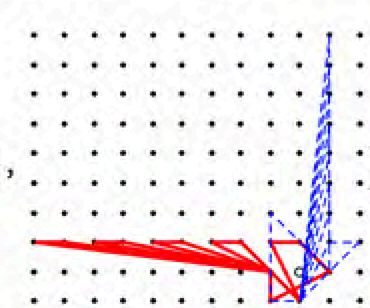
$$\begin{pmatrix}1&2\\1&3\end{pmatrix}, \begin{pmatrix}1&3\\0&-1\end{pmatrix}, \begin{pmatrix}0&-1\\1&4\end{pmatrix}, \begin{pmatrix}1&4\\1&5\end{pmatrix}, \begin{pmatrix}1&5\\0&-1\end{pmatrix}, \begin{pmatrix}0&-1\\1&6\end{pmatrix}, \begin{pmatrix}1&6\\1&7\end{pmatrix}, \begin{pmatrix}1&7\\0&-1\end{pmatrix}, \begin{pmatrix}0&-1\\1&8\end{pmatrix}, \begin{pmatrix}1&8\\1&0\end{pmatrix}, \begin{pmatrix}1&0\\-1&2\end{pmatrix}\},$$

$$, |\tau| = , \{3, 1, 2, 1, -1, 1, 1, -1, 1, 1, -1, 1, 1, -1, 1, -8, 2\},$$

$$, \check{\Xi}= , \{\begin{pmatrix}-1&-2\\1&-1\end{pmatrix}, \begin{pmatrix}-1&1\\0&-1\end{pmatrix}, \begin{pmatrix}1&1\\-2&0\end{pmatrix}, \begin{pmatrix}1&-1\\-1&2\end{pmatrix}, \begin{pmatrix}-1&-1\\0&1\end{pmatrix}, \begin{pmatrix}2&1\\-1&0\end{pmatrix}, \begin{pmatrix}3&-2\\-1&1\end{pmatrix},$$

$$\begin{pmatrix}-1&-3\\0&1\end{pmatrix}, \begin{pmatrix}4&1\\-1&0\end{pmatrix}, \begin{pmatrix}5&-4\\-1&1\end{pmatrix}, \begin{pmatrix}-1&-5\\0&1\end{pmatrix}, \begin{pmatrix}6&1\\-1&0\end{pmatrix}, \begin{pmatrix}7&-6\\-1&1\end{pmatrix}, \begin{pmatrix}-1&-7\\0&1\end{pmatrix}, \begin{pmatrix}8&1\\-1&0\end{pmatrix}, \begin{pmatrix}0&-8\\-1&1\end{pmatrix}, \begin{pmatrix}2&0\\1&1\end{pmatrix}\},$$

$$, \Delta^{\star} \overset{?}{=} , \begin{pmatrix}1&0&-1&0&-2&-3&-1&-4&-5&-1&-6&-7&-1&-8&-9&-1&-1\\0&1&1&-1&1&1&0&1&1&0&1&1&0&1&1&0&-1\end{pmatrix}, \text{Involution}\},$$



$$\}$$