# Through the Looking-Glass, & What Mathematics Lurks There

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# Playbill

*"If you don't know where you're going,*
*it doesn't matter which road you take."*

- Through a Looking Glass:
  *What is mirror duality*
- First Few Fragments of a Fable
  *Why it is worthwhile*
- Forging and Fanciful Framing
  *How mirrors are made*
  *in collaboration with Per Berglund*
- Festivities of a Fine Future
  *Where the foundation are...*

"My mind is open." — Erdős Pál



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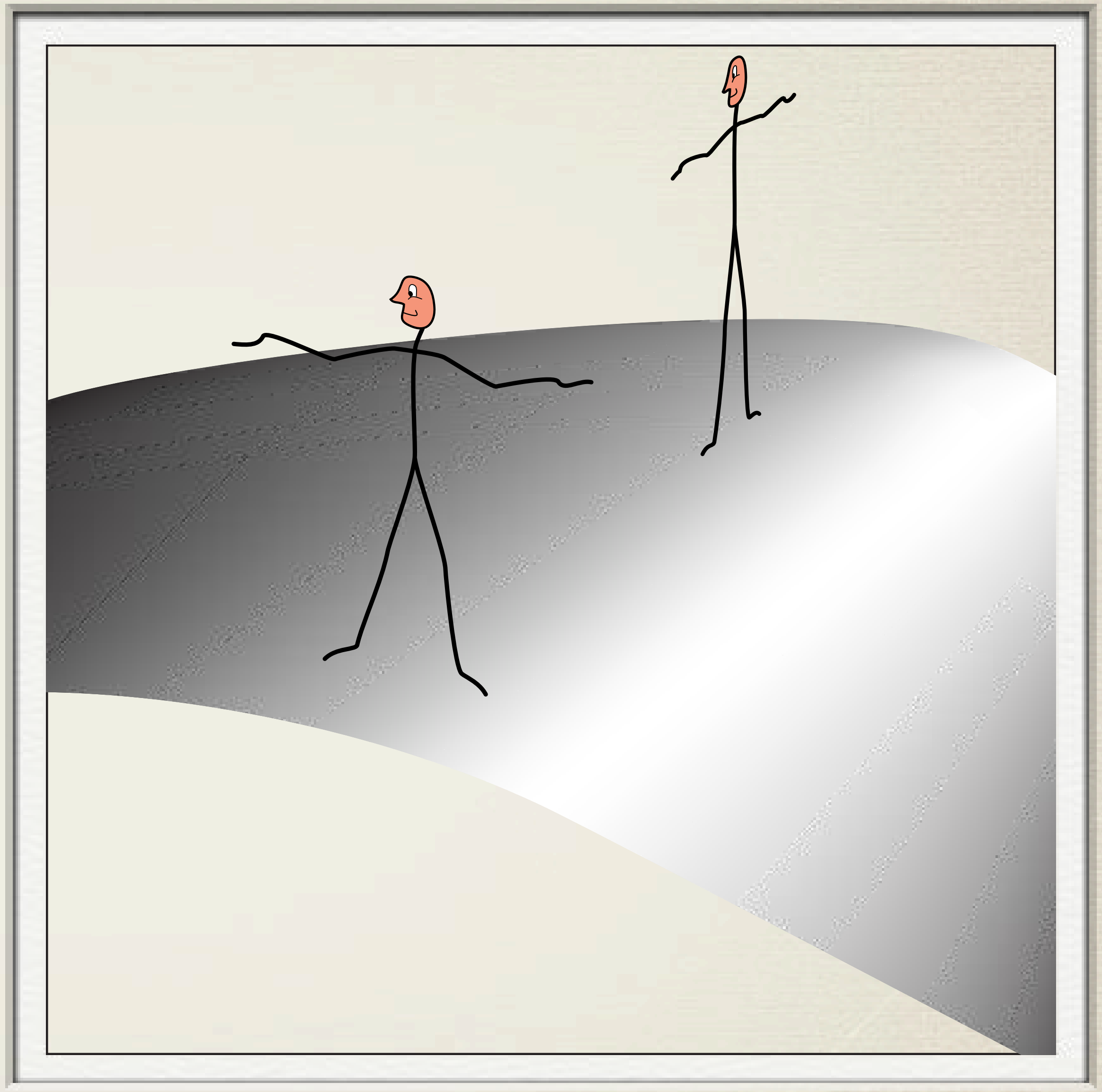
# Mirror Duality in String Theory

# Mirror Duality in String Theory

- Compactifying space-time
  - The surface to the right is…

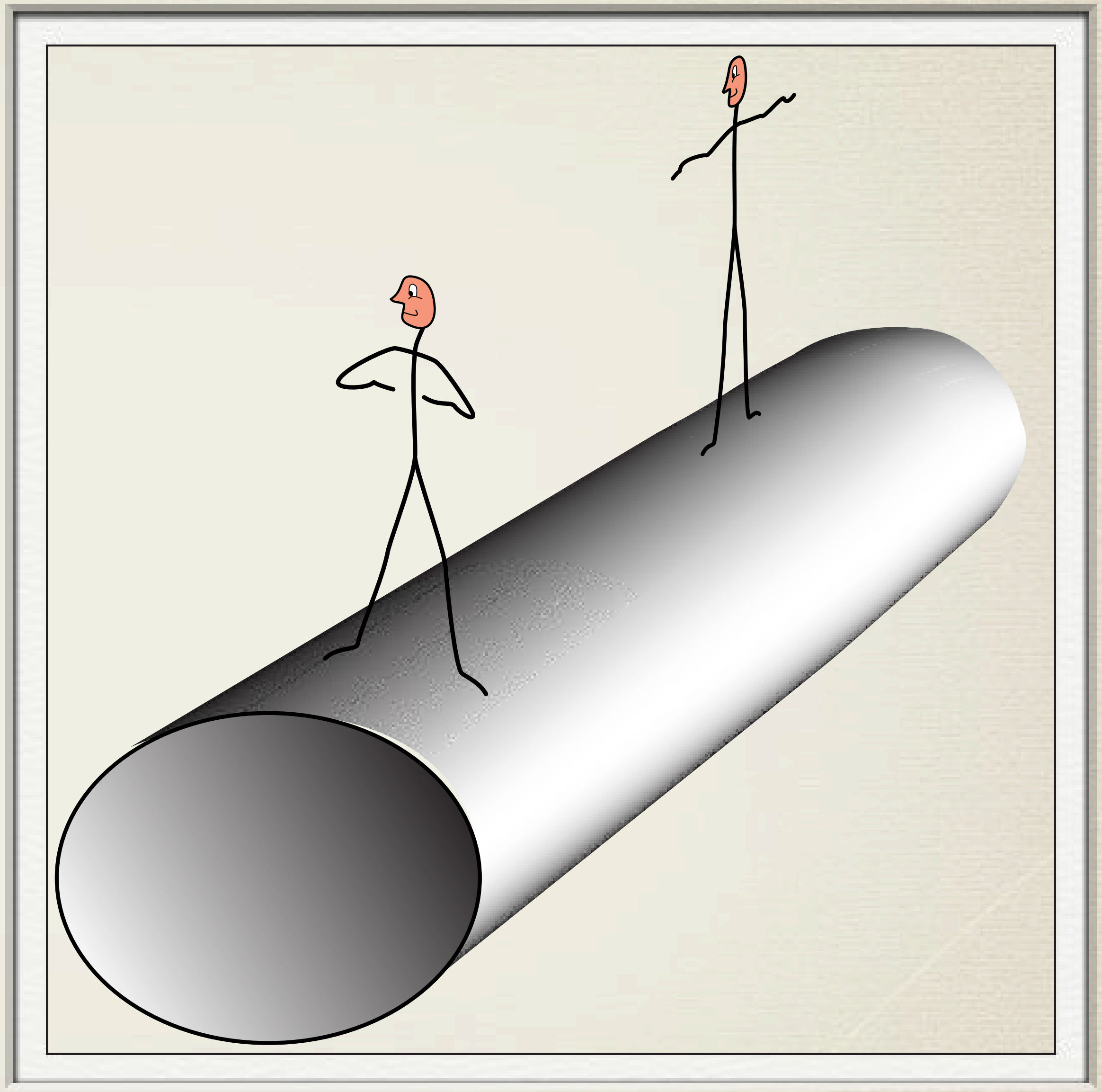
A very 2-dimensional surface, both indefinitely long and indefinitely wide



# Mirror Duality in String Theory

- Compactifying space-time
  - The surface to the right is…

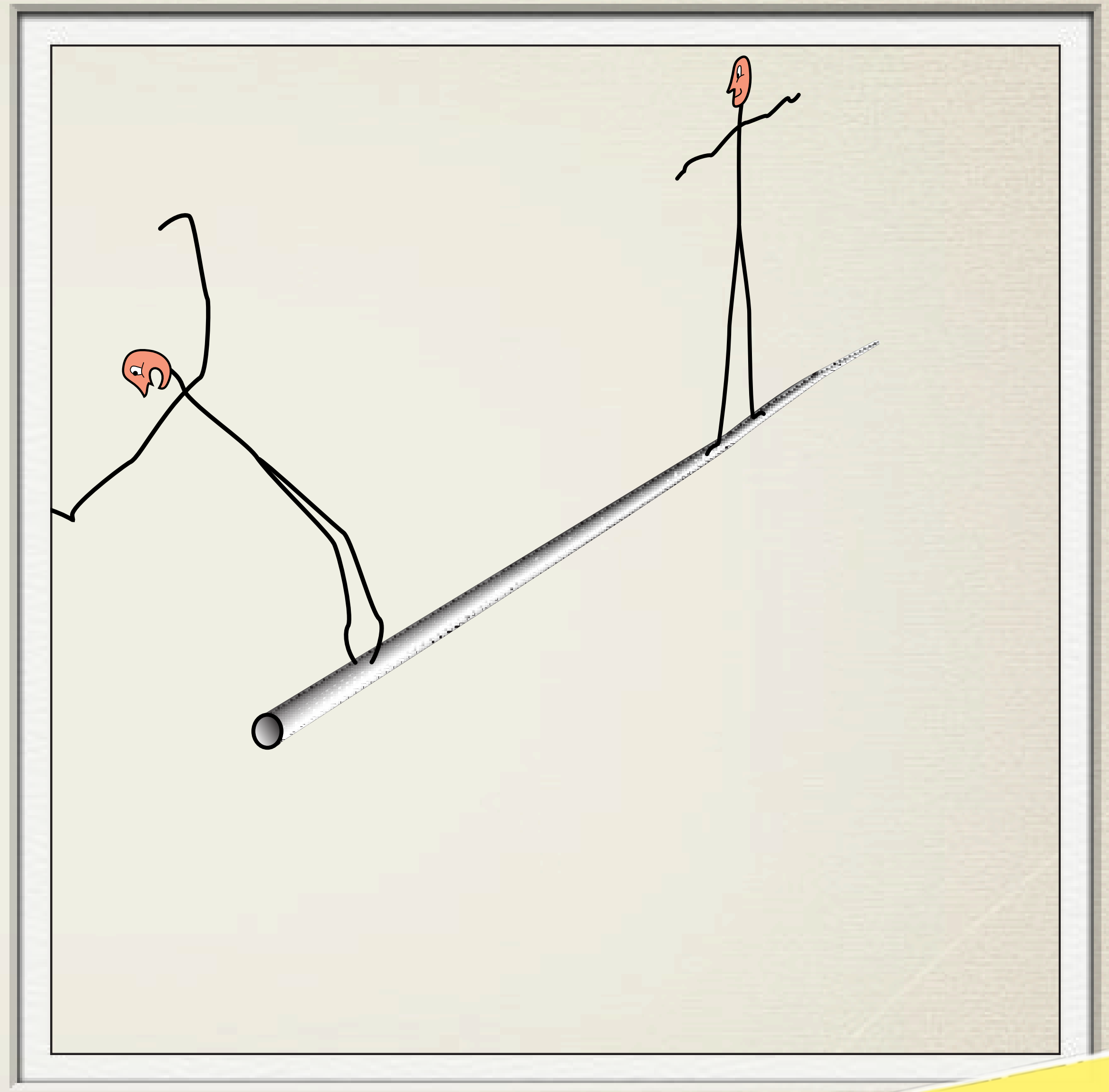
Still, a 2-dimensional surface, indefinitely long and somewhat wide



# Mirror Duality in String Theory

- Compactifying space-time
  - The surface to the right is...

Only a 1-dimensional ***line*** indefinitely long and ***veeery*** thin



- Topologically, it remains $\mathbb{R}^1 \times S^1$, even if the circle
  - is too big to be seen as curved, or
  - is too small to be seen at all

But, strings perceive space differently!



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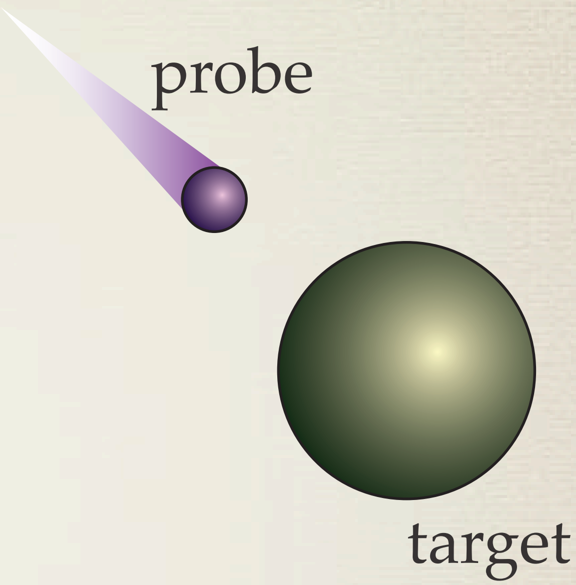
# Mirror Duality in String Theory
*cont'd*

- But… *why do we need strings?*
  - Because we can't see ***points***, that's why:
  - When a probe of energy $E_{\mathrm{p}} = \hbar c/\lambda_{\mathrm{p}} = m_{\mathrm{p}}c^2$, interacts with a target of mass $m_{\mathrm{t}}$…



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# Mirror Duality in String Theory

*cont'd*

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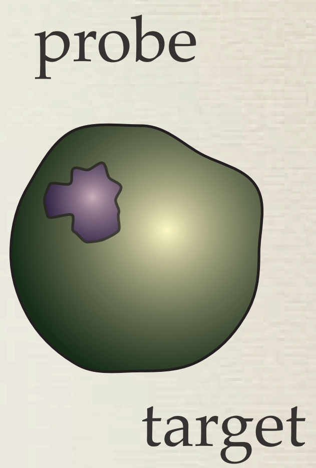
  - When a probe of energy $E_{\rm p} = \hbar c/\lambda_{\rm p} = m_{\rm p}c^2$, interacts with a target of mass $m_{\rm t}$…
  - …during interaction, they form a system with a composite (effective) mass $(m_{\rm t} + \hbar/\lambda_{\rm p}c)$,
  - …from which the escape velocity equals the speed of light at $R_{\rm S} = 2G_{\rm N}(m_{\rm t} + \hbar/\lambda_{\rm p}c)/c^2$.



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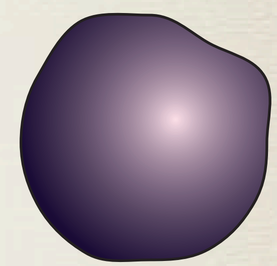
  - …from which the escape velocity equals the speed of light at $R_{\rm S} = 2G_{\rm N}(m_{\rm t} + \hbar/\lambda_{\rm p}c)/c^2$.
  - So, once $\lambda_{\rm p} = R_{\rm S}(= \ell_{\rm P} \sim 10^{-35}{\rm m}$ = "Planck length"), the probe stops seeing "inside" the target.



probe+target

"multipole expansion"

- And strings are the "next best thing,"
- …with significant complications:
  - worldlines *vs*. worldsheets (Riemann surfaces w/marked pts.)

# Mirror Duality in String Theory
*cont'd*

- Indeed, in the multipole expansion, the next "thing" after a point is a dipole—*i.e.*, a stick.
- Except, relativistic "sticks" cannot possibly be rigid.
- Strings have a finite and nonzero tension:

$$T_0 = \frac{1}{2\pi\alpha'\hbar c} = \frac{c^4}{2\pi G_N} \qquad \ell_P = \sqrt{\frac{\hbar G_N}{c^3}}$$

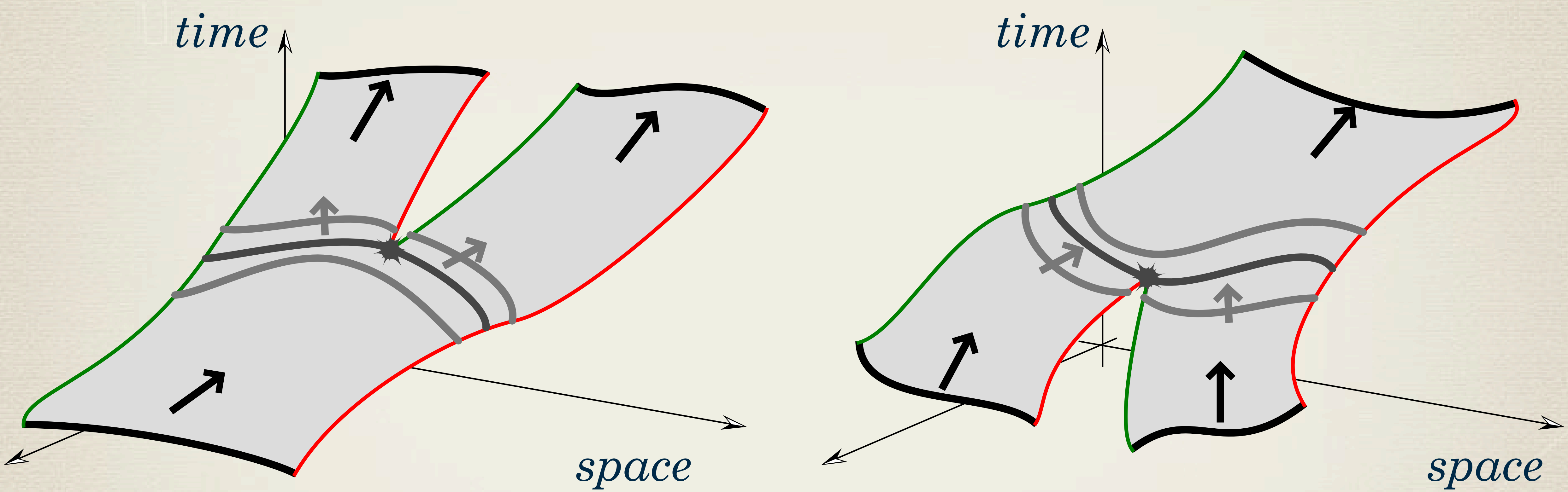
- …and so also a characteristic size, $\ell_S = \ell_P \sim 10^{-35}$m.

branes

- And strings (modeled on mesons) interact:
  - by breaking in two, and
  - by joining ends.

# Mirror Duality in String Theory
*cont'd*

- Something like this:

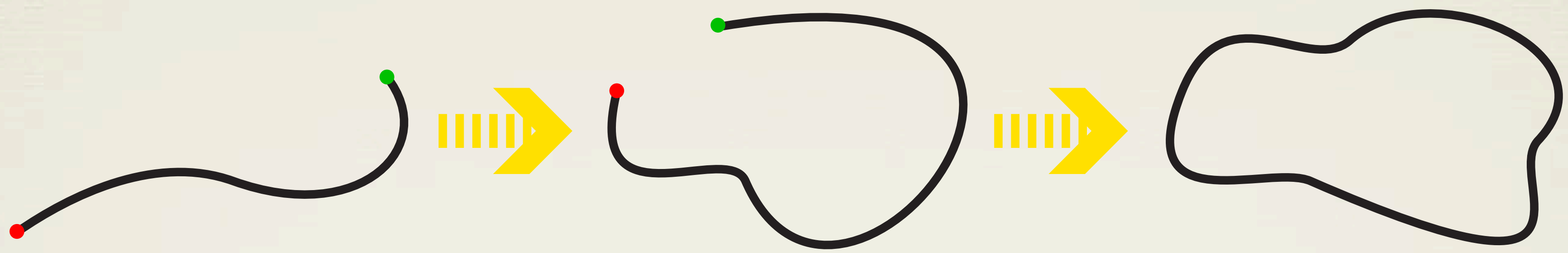



- It is then consistent to have opposite charges at the endpoints. (*In fact, only SO(32) charges will do, but that's technical.*)
- Also: there are always closed (O-like) strings.

# Mirror Duality in String Theory

*cont'd*

- Self-interaction…



- …guarantees: open strings are inconsistent without closed ones, but closed strings are self-consistent.
- Strings vibrate:

$$\alpha' = \frac{G_N}{\hbar\, c^5}$$

$$X^\mu(\tau,\sigma) = \boxed{x^\mu + \frac{p_\mu}{p_-} c\tau} + i\hbar c \sqrt{\frac{\alpha'}{2}} \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \left[ \frac{a^\mu_{n,R}}{n} e^{-2\pi i n \varsigma^+} + \frac{a^\mu_{n,L}}{n} e^{2\pi i n \varsigma^-} \right]$$

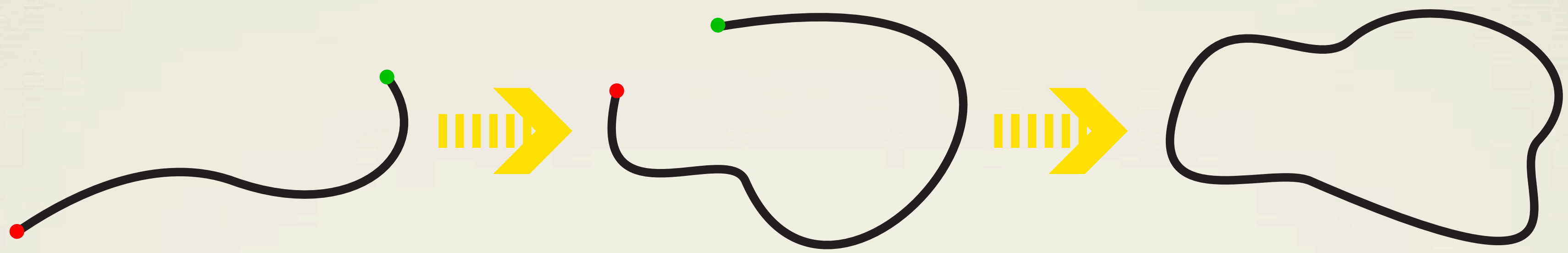
*on-shell, c.m. motion*

where

$$\varsigma^\pm := (\sigma \pm c\tau)/\ell_S, \qquad \text{and} \qquad a^\mu_{-n,R} = (a^\mu_{n,R})^\dagger, \quad a^\mu_{-n,L} = (a^\mu_{n,L})^\dagger.$$

# Mirror Duality in String Theory
*cont'd*

- Self-interaction...



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*oscillations/fluctuations*

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# Mirror Duality in String Theory

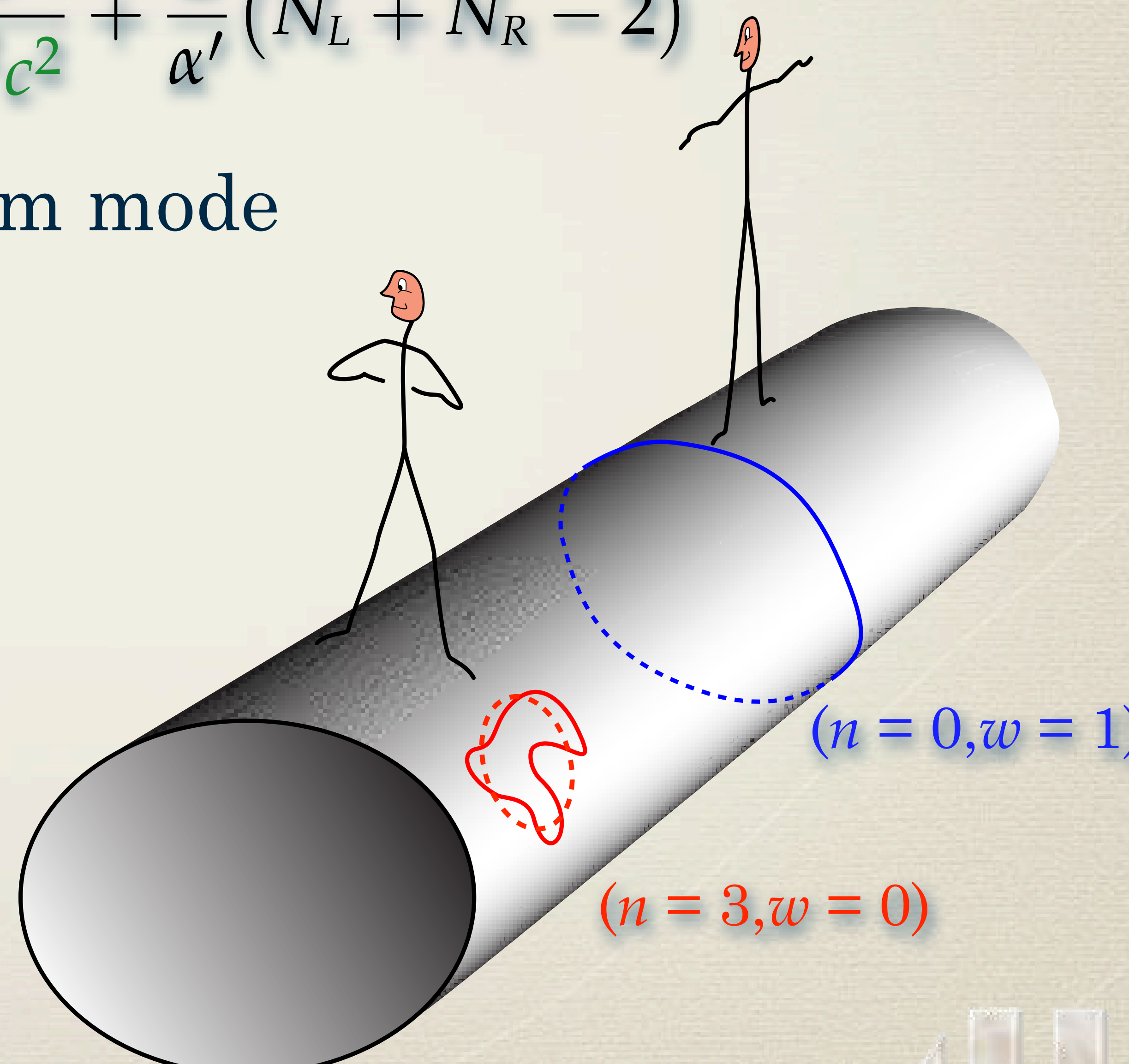
*cont'd*

- The (energy/mass)$^2$ of a mode of string motion then is:

$$\|P_\mu c\|^2 = (mc^2)^2 = \frac{n^2\hbar^2c^2}{R^2} + \frac{w^2R^2}{\alpha'^2\hbar^2c^2} + \frac{2}{\alpha'}(N_L + N_R - 2)$$

- $n$ = (periodic space) momentum mode
- $w$ = string winding mode
- $N_L$, $N_R$ = total number of oscillations
- Small-Large transform.:

$$n \leftrightarrow w \quad \text{and} \quad R \leftrightarrow \frac{\alpha'\hbar^2c^2}{R} = \frac{\ell_s{}^2}{R}$$



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# Mirror Duality in String Theory
*cont'd*

- When a dimension is curled up critically, $R \sim \ell_S$,
  - …then $R \to \ell_S^2/R$ is an actual symmetry.
- Otherwise, it's a small-large ***T-duality***. $(S^1 \times \mathbb{R}^{1,n} \leftrightarrow S^1 \times \mathbb{R}^{1,n})$
- **Clearly**, when more than one dimension is curled up, there are many more ways to ***dualize***.
- When curling up 6 dimensions & preserving 1 susy, one must use a Calabi-Yau manifold.

Compact complex 3-dimensional manifold (typ. algebraic variety) with a Ricci-flat Kähler metric

# Mirror Duality in String Theory
*cont'd*

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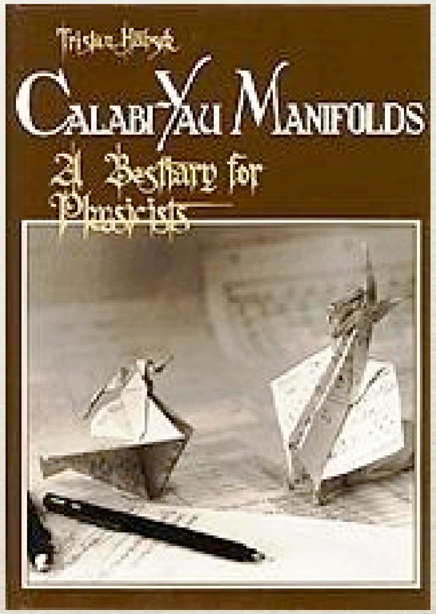
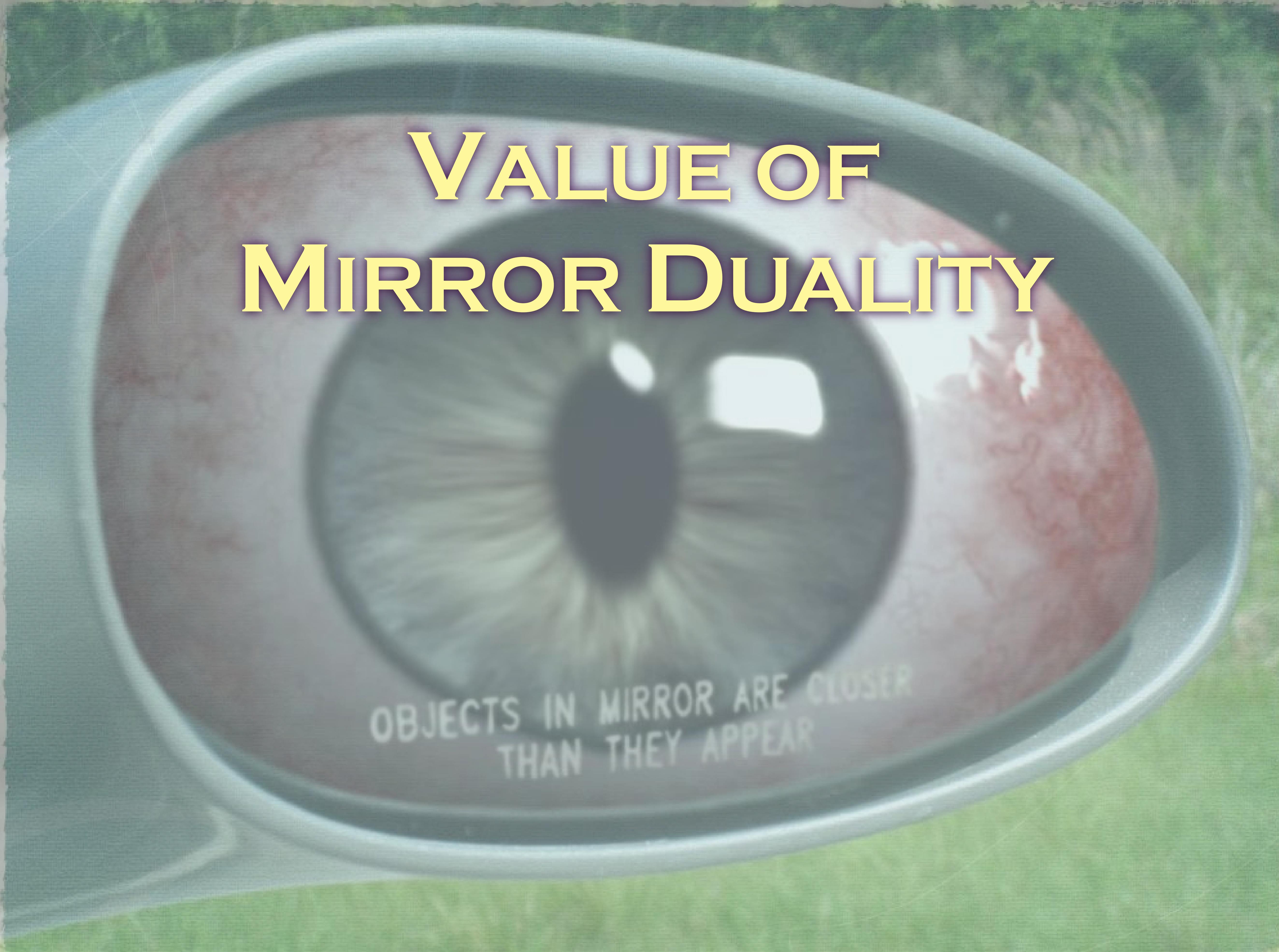
- **Clearly** , when more than one dimension is curled up, there are many more ways to ***dualize***.
- When curling up 6 dimensions & preserving 1 susy, one must use a Calabi-Yau manifold.
- Strominger-Yau-Zaslow (1996): "Mirror duality of Calabi-Yau manifolds is a T-duality"



# Value of Mirror Duality



# Value of Mirror Duality

- First off, ***What*** does mirror duality ***do***?
- When space is curled up on a CY 3-fold $\mathscr{Y}$, the string model in the remaining 3+1-dimensional spacetime has
  - $h^{2,1}$ "generations" of Standard Model matter modes,
  - and $h^{1,1}$ "anti-generation" matter modes,
  - plus some completely charge-less "junk".
- Mirror duality swaps that:
  - $M(h^{2,1}, h^{1,1}) \leftrightarrow W(h^{1,1}, h^{2,1})$,
  - discovered (Greene & Plesser, 1990)

...and we thought 'em to be **bonkers**!

# Value of Mirror Duality

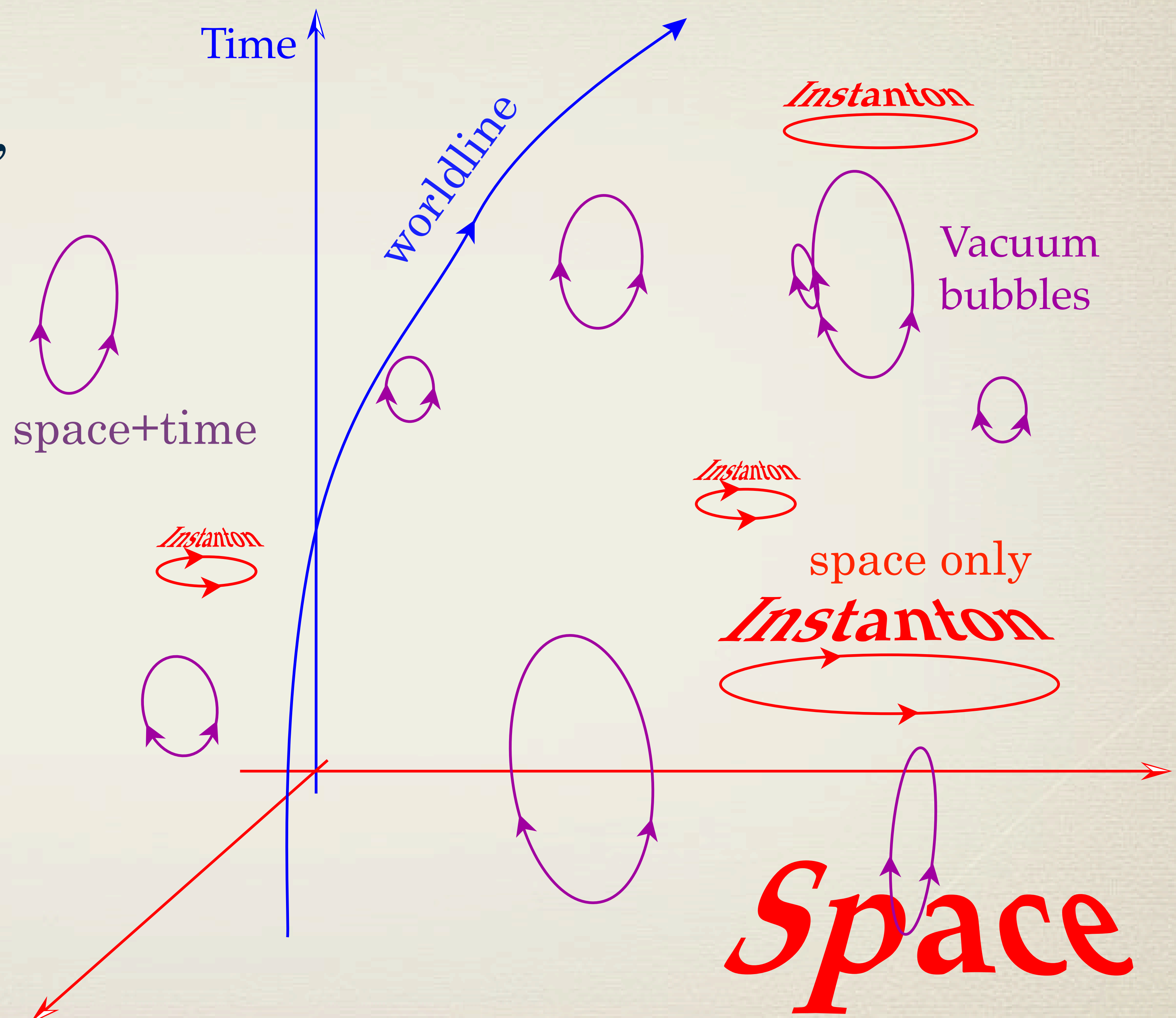
*cont'd*

- Why would this be so strange?
  - Matter: $h^{2,1} = \dim H^1(\mathscr{Y}, \wedge^2 \mathscr{T}_{\mathscr{Y}}{}^*) = \dim H^1(\mathscr{Y}, \mathscr{T}_{\mathscr{Y}})$,
  - Mirror-matter: $h^{1,1} = \dim H^1(\mathscr{Y}, \mathscr{T}_{\mathscr{Y}}{}^*)$
  - "Junk": $= \dim H^1(\mathscr{Y}, \mathrm{End}\,\mathscr{T}_{\mathscr{Y}}) = \dim H^1(\mathscr{Y}, \mathscr{T}_{\mathscr{Y}} \otimes \mathscr{T}_{\mathscr{Y}}{}^*)$
- So, mirror duality seems to map $(\mathscr{Y}, \mathscr{T}_{\mathscr{Y}}, \mathscr{T}_{\mathscr{Y}}{}^*) \leftrightarrow (\mathscr{X}, \mathscr{T}_{\mathscr{X}}{}^*, \mathscr{T}_{\mathscr{X}})$ !?
- Even for CY $n$-folds, $\wedge^n \mathscr{T}_{\mathscr{Y}} = \mathscr{O}_{\mathscr{Y}}$, so $\wedge^p \mathscr{T}_{\mathscr{Y}} = \wedge^{n-p} \mathscr{T}_{\mathscr{Y}}{}^*$,
- …this would imply $\mathscr{T}_{\mathscr{Y}} \leftrightarrow \wedge^{n-1} \mathscr{T}_{\mathscr{X}}$. *A leetle better, but still crazy.*
- In fact, it's crazier still: $H^3(\mathscr{Y}) = \mathbb{C} = H^0(\mathscr{Y}, \wedge^3 \mathscr{T}_{\mathscr{Y}})$
  - $\mathrm{Ring}(H^1(\mathscr{Y}, \mathscr{T}_{\mathscr{Y}}), \star) = \mathrm{Ring}(H^1(\mathscr{X}, \mathscr{T}_{\mathscr{X}}{}^*), \wedge_Q)$
  - Def. $\star$: $\varkappa_{abc} = \int_{\mathscr{Y}} \Omega \wedge \varepsilon(a \wedge b \wedge c)$
  - Def. $\wedge_Q$: $\varkappa_{\alpha\beta\gamma} = \int_{\mathscr{X}} \alpha \wedge \beta \wedge \gamma + \underbrace{\Sigma_C\, e^{-(\int_C J)} (\int_C \alpha) (\int_C \beta) (\int_C \gamma)}_{\text{"instanton contributions"}}$, $C \in H_2(\mathscr{X})$

# Value of Mirror Duality

*digression*

- Instantons?
- Vacuum “bubbles” = virtual particles
  - pop out of the vacuum,
  - propagate,
  - vanish into the vacuum.
- Especially interesting when noncontractible!

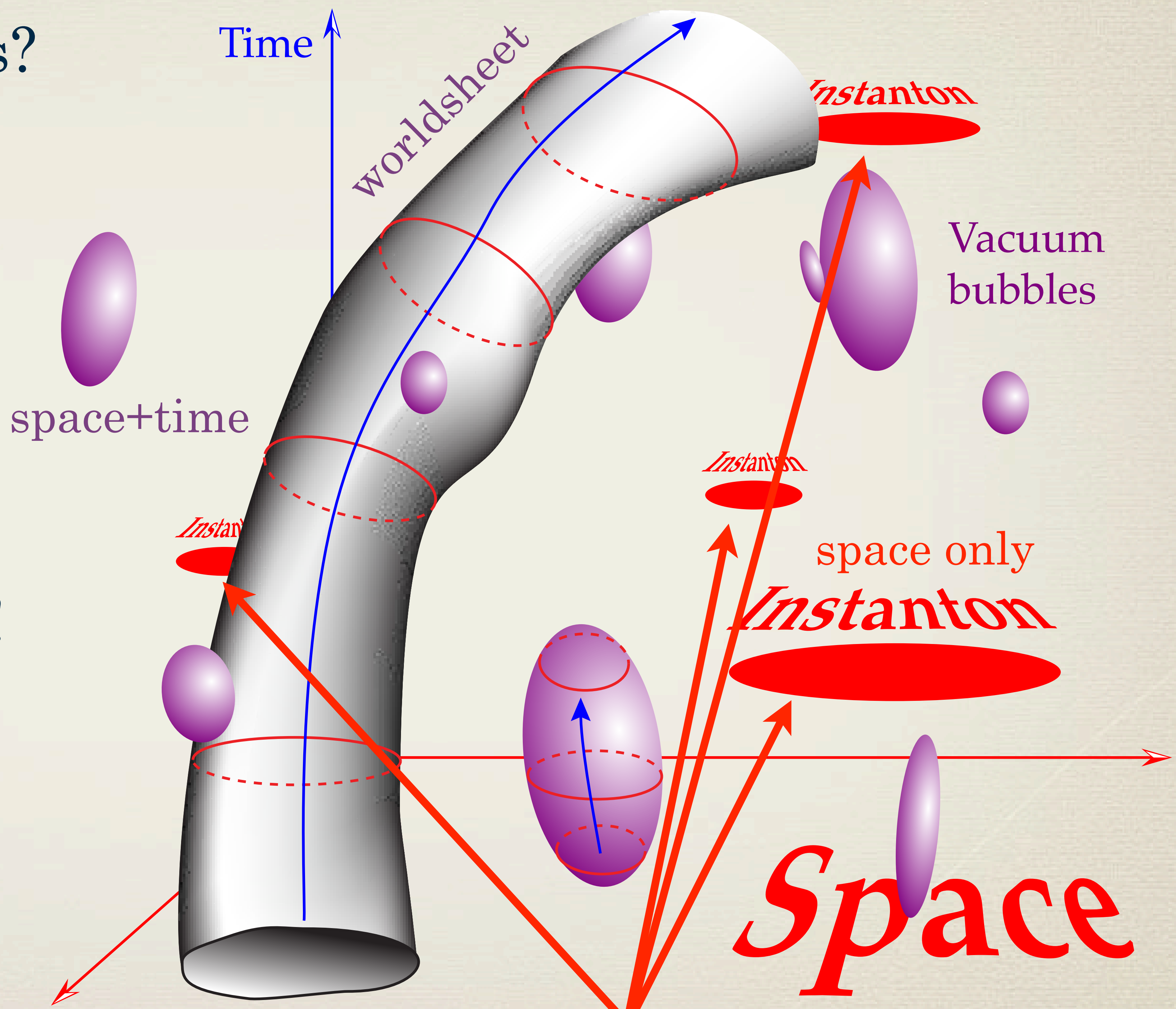



Except, $\pi_1(\mathscr{Y})$ is discrete.

# Value of Mirror Duality

*digression cont'd*

- String instantons?
- Vacuum bubbles = virtual strings
  - pop out of the vacuum,
  - propagate,
  - vanish into the vacuum.
  - Must be bosons!
- Bose-Einstein distribution!
- Especially interesting when noncontractible…




"wrap" nontrivial 2-cycles in the CY 3-fold



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# Value of Mirror Duality

*cont'd*

- So Candelas, de la Ossa, Parkes and Green worked out a detailed example (1991).

Note the palindromic year!

- One CY 3-fold, *M*, is:

$$z_1^{\,5} + z_2^{\,5} + z_3^{\,5} + z_4^{\,5} + z_5^{\,5} - 5\psi\, z_1 z_2 z_3 z_4 z_5 = 0,$$

$$(z_1, z_2, z_3, z_4, z_5) \simeq (\lambda z_1, \lambda z_2, \lambda z_3, \lambda z_4, \lambda z_5)$$

- where $\lambda \neq 0$. This is "$\mathbb{C}P^4[5]$."
- Symmetries: $(z_1, z_2, z_3, z_4, z_5) \cong (\omega^a z_1, \omega^b z_2, \omega^c z_3, \omega^d z_4, \omega^e z_5)$, where $\omega^5 = 1$, and so $(a,b,c,d,e)$ are taken (mod 5).
- Turns out, its mirror model may be constructed as $W = M/(\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5)$; ($h^{1,1}$=101, $h^{2,1}$=1: $\psi$)



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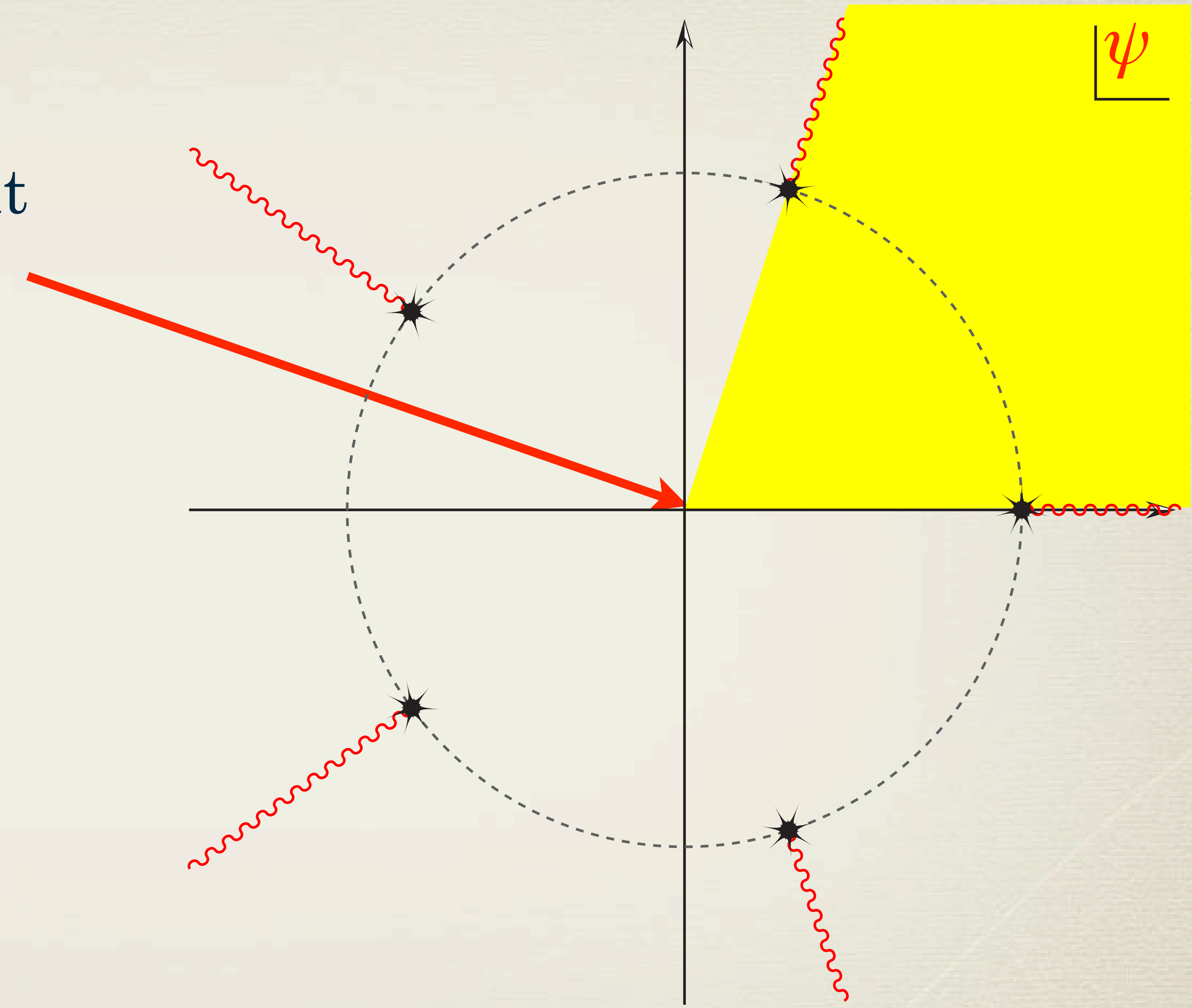
# Value of Mirror Duality

*cont'd*

- So, in the $\psi$-space:
  - $\psi = 0$ is the "Fermat quintic," $\Sigma_i z_i^5 = 0$;



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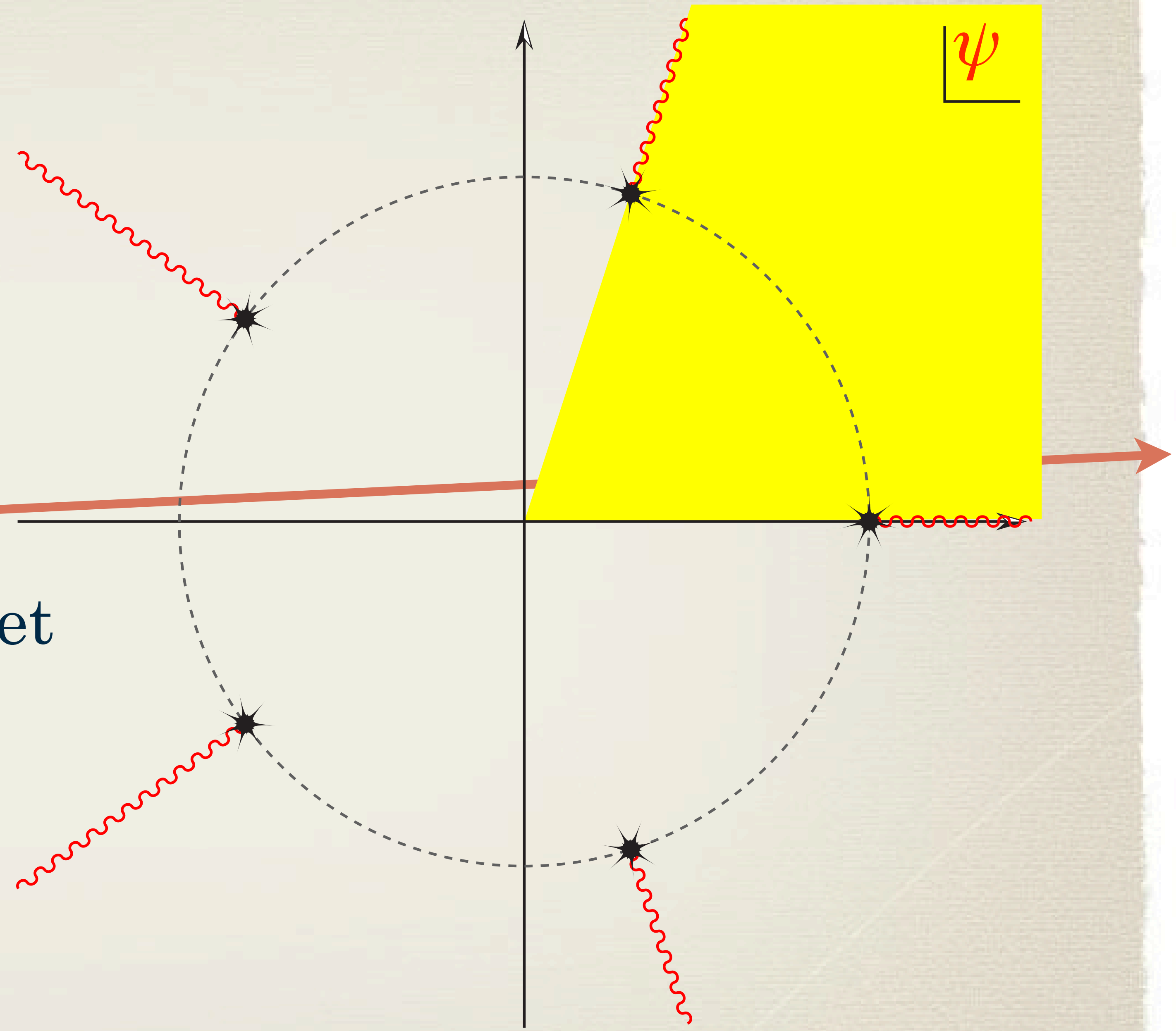
# Value of Mirror Duality

*cont'd*

- So, in the $\psi$-space:
  - $\psi = 0$ is the "Fermat quintic," $\sum_i z_i^5 = 0$;
  - $\psi = \infty$ is the "*quint-ogicon*", $\prod_i z_i = 0$:
  - ($5\ \mathbb{C}^3$'s meet in $10\ \mathbb{C}^2$'s meet in $10\ \mathbb{C}^1$'s meet in $5\ \mathbb{C}^0$'s);



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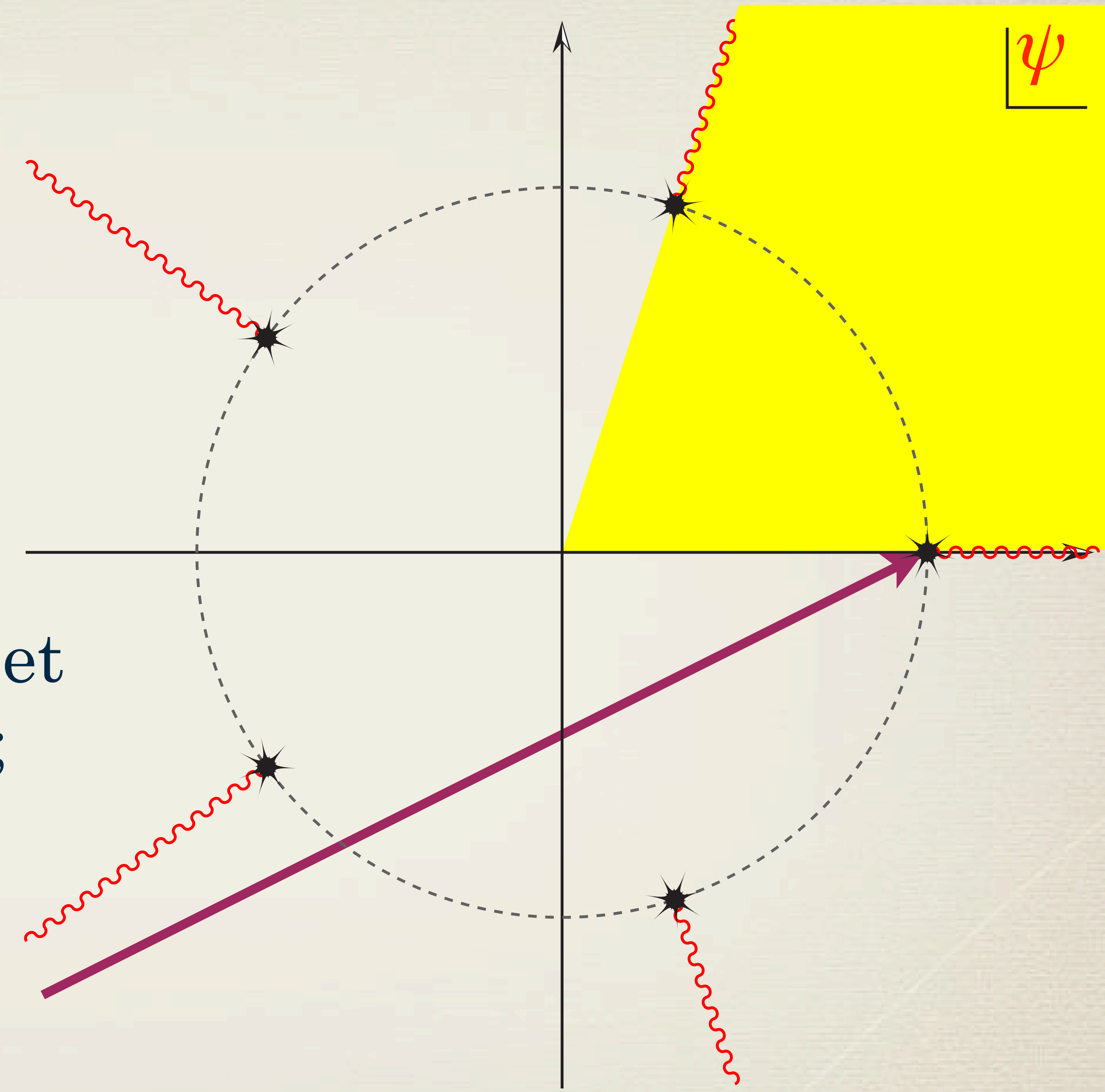
  - $\psi = 1$ is the "quintic *conifold*": $f(z) = 0 = \mathrm{d}f(z)$…



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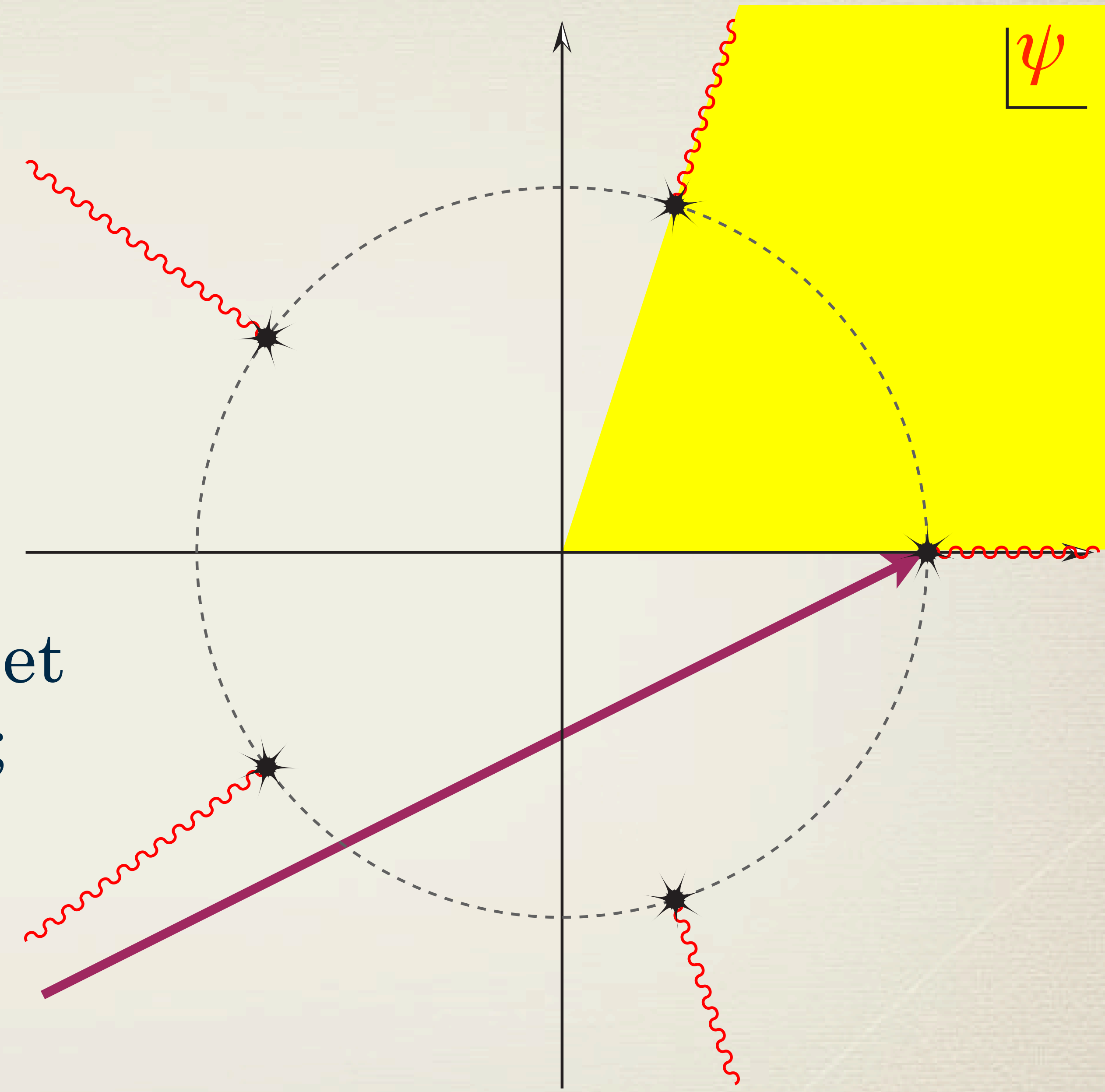
  - $\psi = 1$ is the "quintic *conifold*": $f(z) = 0 = df(z)$…
  - …which is *singular**. [http://en.wikipedia.org/wiki/Conifold]




* $df = \sum_i (\partial_i f)\, dz^i$, so that $\{df = 0\} \Leftrightarrow \{(\partial_1 f, \partial_2 f, \ldots, \partial_5 f) = 0\}$.





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# Value of Mirror Duality

*cont'd*

- Long story short, physicist *compute*. And, have found [COGP] that the Yukawa coupling $\varkappa_{ttt}(\psi)$ for $W$ may be expressed as a power series

$$5+\sum_{k=1}^{\infty}\frac{n_k k^3 e^{2\pi i k t}}{1-e^{2\pi i k t}} = 5+2875\,e^{2\pi i t}+\ldots$$

RECOGNIZE THIS?

$$t := -\frac{5}{2\pi i}\left\{\log(5\psi)-\frac{1}{\varpi_0(\psi)}\sum_{m=1}^{\infty}\frac{(5m)!}{(m!)^5(5\psi)^{5m}}\left(\psi^{(0)}(1+5m)-\psi^{(0)}(1+m)\right)\right\}$$

- This tells the interaction strength for *every* model in this 1-parameter family.
- The $n_k$ give the number of degree-$k$ embeddings of $\mathbb{C}P^1$'s ($S^2$'s – *string instantons*) in $W$.

# Value of Mirror Duality

*cont'd*

- For a 1-parameter family of varieties $W$: $f(z)+\psi g(z) = 0$,
- Atiyah-Bott-Gårading: $\Omega(\psi) := \operatorname*{Res}_{W\subset \mathbb{C}P^4}\left[\frac{(z\mathrm{d}^4z)}{f(z)+\psi g(z)}\right] = \int_{\Gamma_3}\frac{(z\mathrm{d}^4z)}{f(z)+\psi g(z)}$
- Bryant-Griffiths: $\frac{\partial\Omega(\psi)}{\partial\psi} = K(\psi)\Omega(\psi)+\varphi(\psi), \quad \varphi(\psi)\in H^1(W,\mathcal{T}_W)$
  - As a residue integral, $\varphi(\psi)$ is called a "period" of $\Omega(\psi)$.
- The 3rd derivative is the leading contribution to $\varkappa_{ttt}(\psi)$…
- …the full result is obtained through a hypergeometric transformation……b/c the $\varphi(\psi)$ satisfy a generalized HGEq.

expand in $\psi$

- Fall 1991 Workshop @ MSRI:
  - COPG list the #s of degree-$d$ $\mathbb{C}P^1$'s $W$, for $d$ = 1, 2, 3, … 10.
  - The first two agree with "classic" enum. algebraic geometry…
  - The d = 3 number had just been computed. And differed.

# Value of Mirror Duality

*cont'd*

- This fresh, new result, by Ellingsrud & Stromme, obtained for $d = 3$, was based on Hilbert schemes and also involved a computer program for a portion of the computation.
- Mathematicians (all experts in algebraic geometry) at the workshop (Yau, Singer, Katz, Clemens, Morrison…) had "no reason to trust physics, but all the reason to trust math."

Oh well…



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*cont'd*

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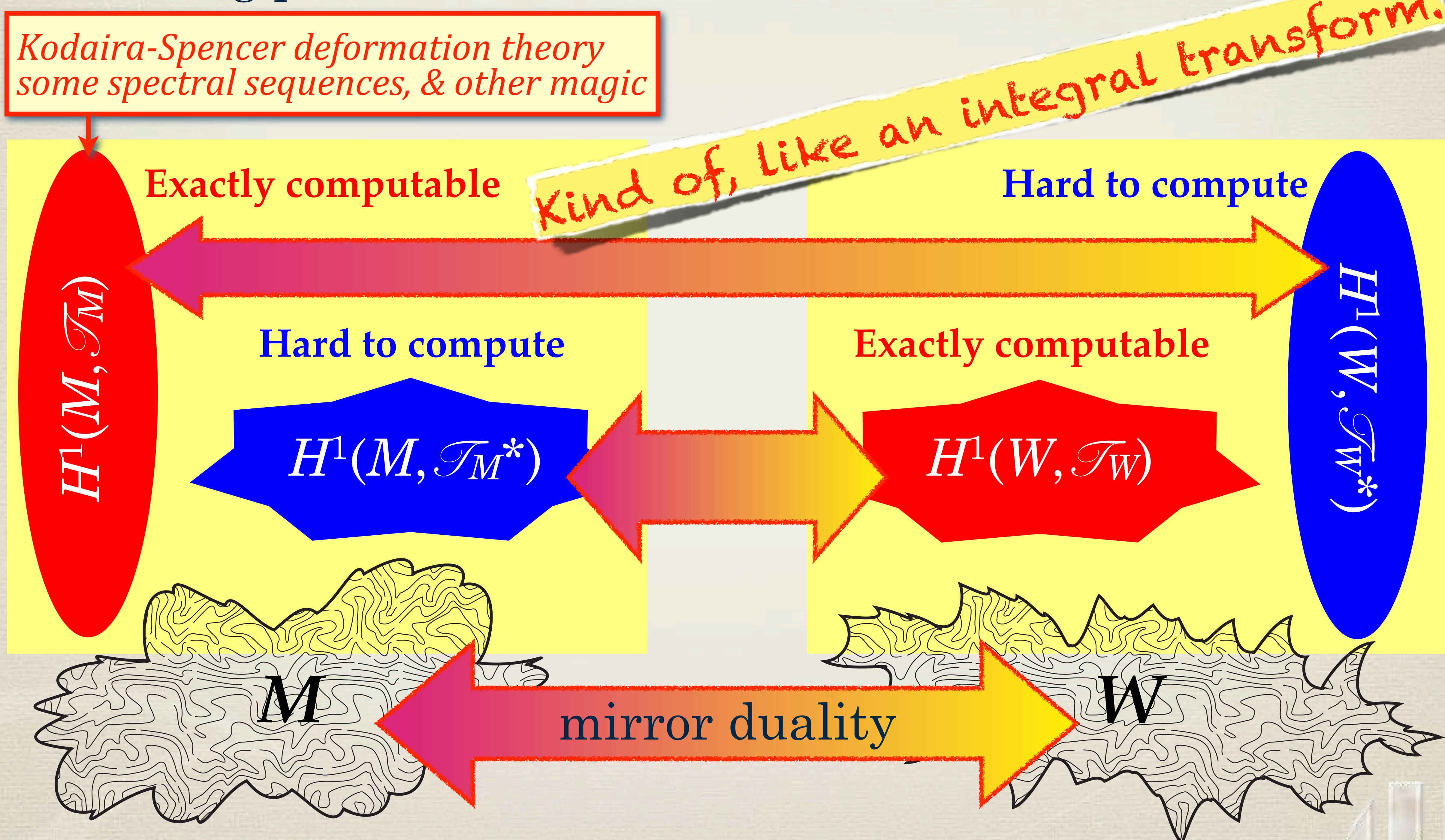
- Two weeks after the workshop, Ellingsrud & Stromme reported to have squashed a bug in their code, after which their result……agreed with the physics result.
- Within another week, Morrison generalized the computation to five other hypersurfaces in weighted projective spaces.

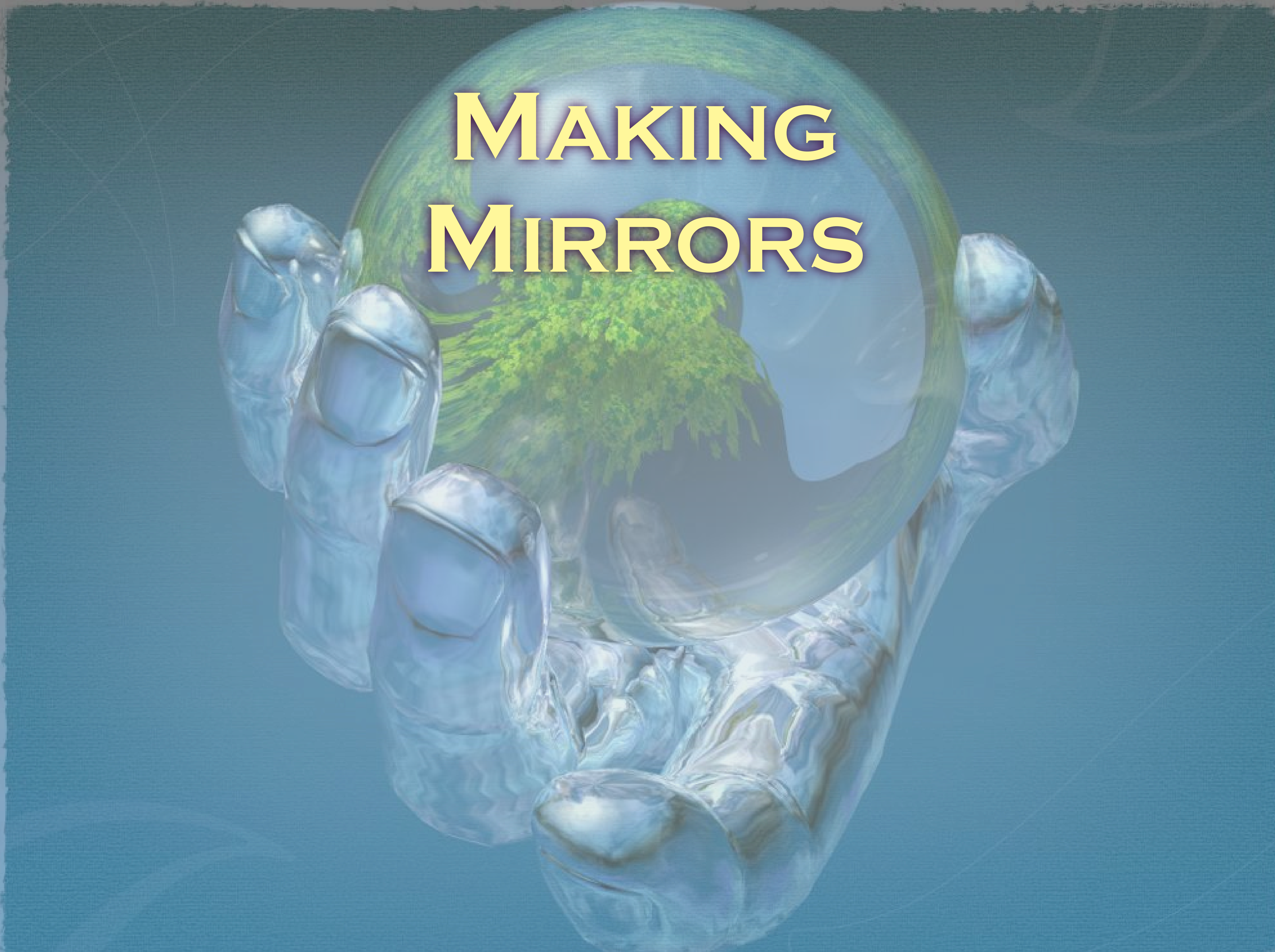
# Value of Mirror Duality

*cont'd*

- The "big picture":



# Making Mirrors

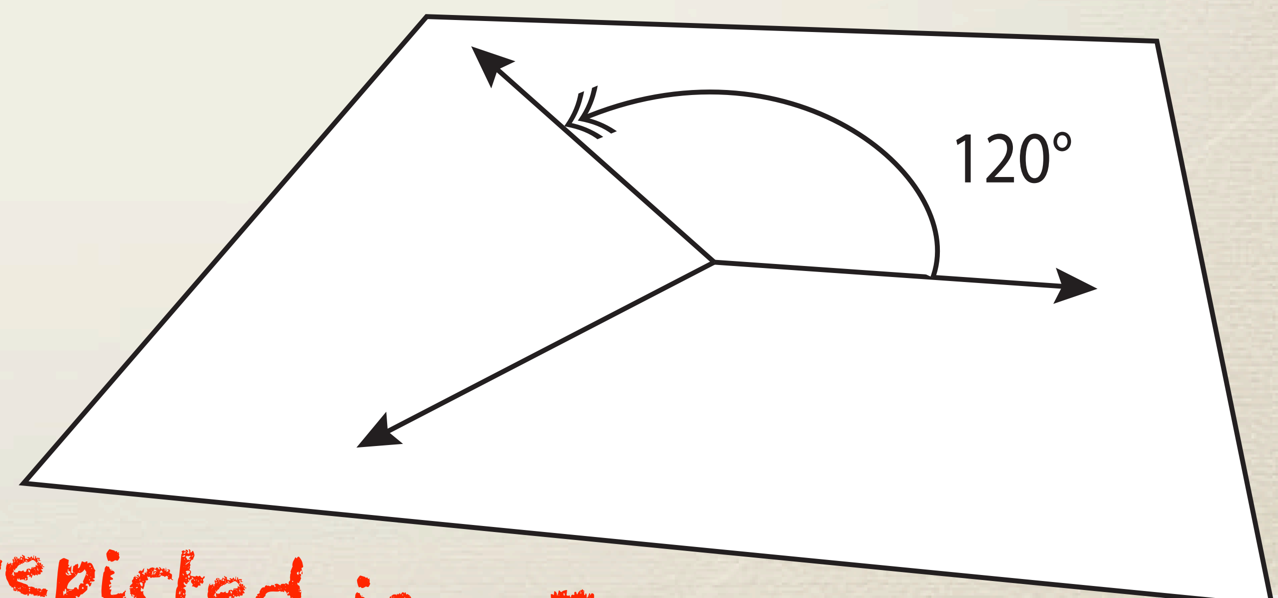


*w/Per Berglund*

# How Mirrors are Made

- So, how does one ~~find~~ make the mirror dual?
- In the mirror pair of quintics, if $M{:=}\{z\in\mathbb{CP}^4, f(z)=0\}$, then $W{:=}M/(\mathbb{Z}_5\times\mathbb{Z}_5\times\mathbb{Z}_5)$. ☺
- But, this is not true in general. ☹
- So we take $f(z) = z_1^5+ z_2^5+ z_3^5+ z_4^5+ z_5^5$,
  - analyze (*disect*) how does $\mathbb{Z}_5\times\mathbb{Z}_5\times\mathbb{Z}_5$ act,
  - where do mirror *modes* come from,
  - how do they interact?

quotient ("quantum") symmetry




Depicted is a $\mathbb{Z}_3$ quotient.

*w/Per Berglund*

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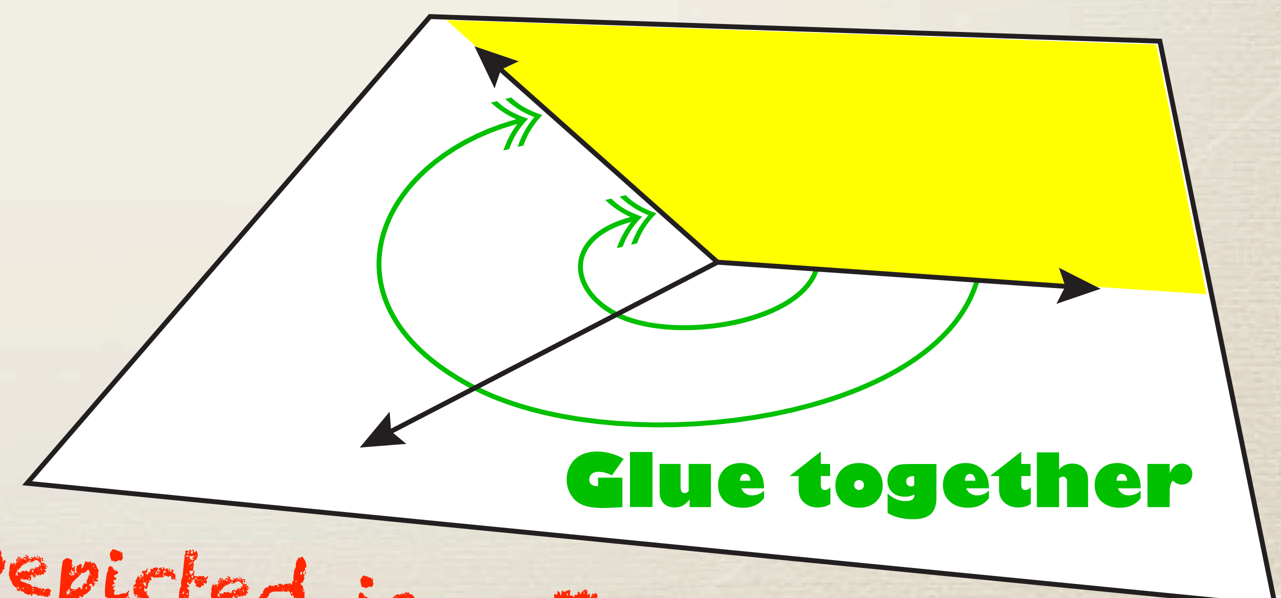


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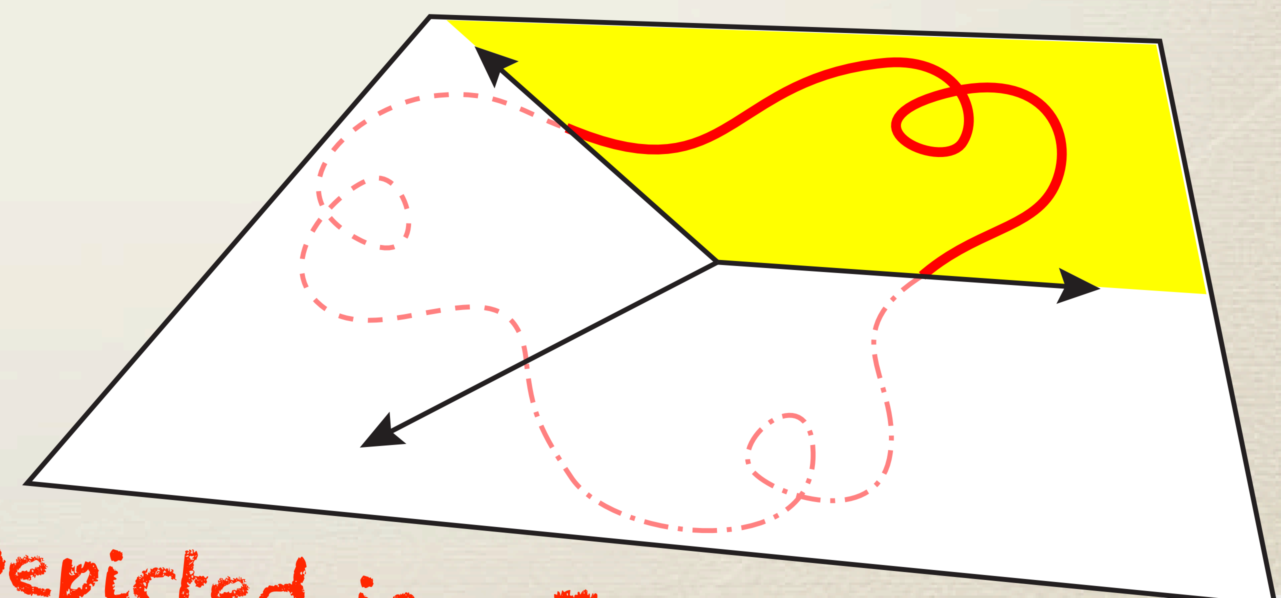
# How Mirrors are Made

- So, how does one ~~find~~ make the mirror dual?
- In the mirror pair of quintics, if $M := \{z \in \mathbb{CP}^4, f(z) = 0\}$, then $W := M/(\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5)$. ☺
- But, this is not true in general. ☹
- So we take $f(z) = z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5$,
  - analyze (*disect*) how does $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$ act,
  - where do mirror *modes* come from,
  - how do they interact?

quotient ("quantum") symmetry



Depicted is a $Z_3$ quotient.



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*w/Per Berglund*

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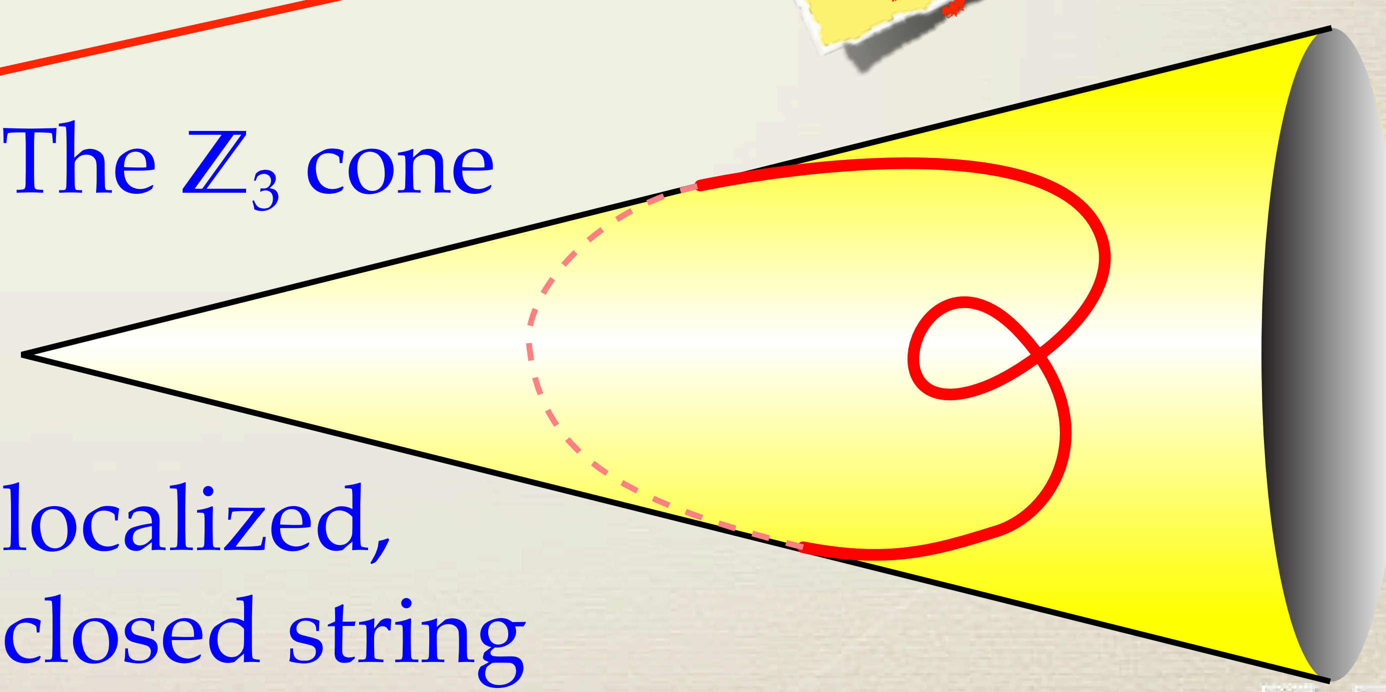
quotient ("quantum") symmetry

The $\mathbb{Z}_3$ cone



localized, closed string

*w/Per Berglund*

# How Mirrors are Made

*cont'd*

- Remember Atiyah-Bott-Gårading? Generalize.
- If $W = M/D$, W contains fixed subsets, $S_i$, of the various subgroups of $D$.
- Construct residue forms on each such (homology equivalence class of) subsets…
- …and verify that the set is a complete basis of $H^1(W,\mathscr{T}_W)$ and $H^1(W,\mathscr{T}_W{}^*)$—even $H^1(W,\mathrm{End}(\mathscr{T}_W))$, when worked out!
- Given these explicit representatives, compute triple products that are $D$-invariant.
  - (If they were not, they wouldn't represent a physical interaction measurement.)



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# How Mirrors are Made

*cont'd*

- String modes (that's all matter and all force fields!):
  - from closed strings (as before), *free-roaming*
  - from open strings that close via quotient symmetry. *localized*
- Find a procedure so that the total Hilbert spaces
  - $\mathscr{H}(M; \textbf{\textit{Sym}}_M) = \mathscr{H}(W; \textbf{\textit{Sym}}_W)$, with states
    - symmetric (*identified*) w.r.t. ***Sym***,
    - generated by $G$, if $W = X/G$.
- Ha!

Find $X$ with $\textbf{\textit{Sym}}_X = G{\times}Q$, so:
$M = X/Q$, with $\textbf{\textit{Sym}}_M = G$, &
$W = X/G$, with $\textbf{\textit{Sym}}_W = Q$.



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# How Mirrors are Made

*cont'd*

- So, how to find $X$ with $\boldsymbol{Sym}_X = G{\times}Q$?
- By analyzing "building blocks"
  - $z^k$,
  - $z_1{}^k + z_1{\cdot}z_2{}^m + \ldots + z_{p-1}{\cdot}z_p{}^n$,

  } Use Arnold's classification of singularities and their smoothing.
- PB & TH find, need to *transpose* a minimal polynomial:
- If $f(z) = \sum_i \prod_j z_i{}^{a(i,j)}$, then $f^{\,T}(w) = \sum_i \prod_j w_i{}^{a(j,i)}$, in $W\mathbb{C}P^n$'s
  - If $f(z) = 0$ and $f^{\,T}(w) = 0$ hypersurfaces are smooth,
  - then, $\boldsymbol{Sym}[f(z)] = \boldsymbol{Sym}[f^{\,T}(w)] = P{\times}G{\times}Q$.
  - then, $M{:=}\{f(z) = 0\}\,/\,P{\times}G$ has $\boldsymbol{Sym}_M = Q$,
  - and $W{:=}\{f^{\,T}(w) = 0\}\,/\,P{\times}Q$ has $\boldsymbol{Sym}_W = G$.

  } mirror duals



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# How Mirrors are Made

*cont'd*

- This is what one might call an experimental proof.
- That is, a proof in ***experimental mathematics***.
- That is, within a class of constructions:
  - most general, known, at the time,
  - define a large class of cases (16 × *unknown**),
  - and prove the duality relation rigorously.
- (It suffices to show that the states in the Hilbert spaces for candidate mirror duals transform in complementary ways w.r.t. $P \times G \times Q$.)

Remember Wigner-Eckardt's theorem from QM?

I'll tell you anyway.

*Eventually, a computer search (by others) found many thousands…



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# How Mirrors are Made

*cont'd*

- Wigner-Eckardt Theorem, for $SU(2)$
  - Given three representations, $R_1$, $R_2$, $R_3$
  - each with a basis, $\{v_{1i}\}$, $\{v_{2j}\}$, $\{v_{3k}\}$
  - the (projected) triple product $R_1 \otimes R_2 \otimes R_3 \to \mathbb{C}$ factorizes:
  - $(v_{1i}, v_{2j}, v_{3k}) = (v_{1i}, v_{2j}, v_{3k}) \cdot (R_1, R_2, R_3)$

Clebsh-Gordan coeff.
depends on the $U(1) \subset SU(2)$
easy to compute & tabulate

"reduced matrix element"
depends only on the $SU(2)$
when $\neq 0$, harder but common

  - A generalization exists for bigger groups
  - …but we only needed relatively simple discrete groups
  - …which are straightforward. ☺

*w/Per Berglund*

# How Mirrors are Made

*cont'd*

- And then…

...silence...


# Finding One's Way

# Where the Foundations Are

- And then, out of the blue...

...16 years later: (old enough for a driver's license)

ALESSANDRO CHIODO AND YONGBIN RUAN

http://arxiv.org/abs/0908.0908v2

ABSTRACT. We prove the *classical mirror symmetry conjecture* for the mirror pairs constructed by Berglund, Hübsch, and Krawitz. Our main

MARC KRAWITZ

http://arxiv.org/abs/0906.0796v1

ABSTRACT. In this article, we study the Berglund–Hübsch transpose construction $W^T$ for invertible quasi-

**BERGLUND-HÜBSCH MIRROR SYMMETRY VIA VERTEX ALGEBRAS**

http://arxiv.org/abs/1007.2633v2

Cool title, dont you think?

# Where the Foundations Are

*cont'd*

**BERGLUND-HÜBSCH MIRROR SYMMETRY VIA VERTEX ALGEBRAS**

LEV A. BORISOV

ABSTRACT. We give a vertex algebra proof of the Berglund-Hübsch duality of nondegenerate invertible potentials. We suggest a way to unify it with the Batyrev-Borisov duality of reflexive Gorenstein cones.

http://arxiv.org/abs/1007.2633v2

from Lev A. Borisov's web-site

- The proof by Lev A. Borisov (Math Dept. @ Rutgers U.) uses the algebra of *vertex operators*. These represent the worldsheet localization of the states in (Fock-)Hilbert space of the given string theory model.
- Borisov also pinpoints the key structural differences between the BH-construction and the previous, "standard" construction of the previous 16 years, called the… …the Batyrev-*Borisov* construction.
- This permits a (now studied) unification of these two approaches.



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# Where the Foundations Are

*cont'd*

- P.Berglund and T.H.: [arXiv:hep-th/9201014] uses:
  - The full $\mathscr{H} = \oplus_r \mathscr{H}_r$ ; $r$ counts the rep's of the full symmetry
  - $A_4$: $z^5 = 0$ clearly has a $\mathbb{Z}_5$ symmetry; $A_k \leftrightarrow A_k/\mathbb{Z}_{k+1}$. *D. Gepner Z. Qiu*
  - $D_k$: $x^{k-1} + xy^2 = 0$ has a $\mathbb{Z}_{2(k-1)}$ symmetry; $D_k \leftrightarrow D_k/\mathbb{Z}_{2(k-1)}$. *P. Berglund T.H.*
  - Now tensor such "building blocks" & projectivize.
- S.Tanabe: [arXiv:math/0506354] verifies: periods of Ω, monodromy data, Poincaré polynomials…
  - no surprise: P.B. & T.H. [hep-th/9411131] provides Atiyah-Bott-Gårding type residues for $H^{p,q}(f^{-1}(0))$
  - The "$\oplus_q H^{q,q}(M)$" ring: K.Intrilligator & C.Vafa [*Nucl. Phys.* **B339** (1990) 95–120]
  - The "$\oplus_q H^{q,n-q}(M)$" ring: H.Fan, T.Jarvis Y.Ruan & E.Witten [arXiv:math/0409434], [arXiv:0712.4021], [arXiv:0712.4025]

# Where the Foundations Are

*cont'd*

- M.Krawitz: [arXiv:0906.0796] proof at the level of Frobenius algebras, & ⇛ Arnold's "strange duality"!
- A.Chiodo and Y.Ruan.: [arXiv:0908.0908], proof for FJRW & cohomology rings & for CY orbifolds (discrete quotients)
- L.A.Borisov: [arXiv:1007.2633] proof for vertex algebras & relation to the 2-decade-standard Batyrev-Borisov constr.
- C.F.Doran and R.S.Garavuso: [arXiv:1109.1686] relation of the above to the Hori-Vafa construction
- Note:
  - The "$\oplus_q H^{q,q}(M)$" ring markedly uses the Kähler class
  - The "$\oplus_q H^{q,n-q}(M)$" ring markedly uses the $\Omega$ class
  - T.Hübsch and S.-T.Yau: *Mod. Phys. Lett.* **A7**(1992)3277–3289 identifies the mirror of the classic Lefschetz $SL(2,\mathbb{C})$ action

classic Lefschetz's $SL(2)$

see also: T.H.: *Mod. Phys. Lett.* **A16** (2001) 663–671, hep-th/9903114



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# Where the Foundations Are
*cont'd*

- And the moral to this story?

"My mind is open." — Erdős Pál

If you do not expect the unexpected,
you will not find it.

— Aristotle

Only dead fish swim with the stream.

— Thomas Malcolm Muggeridge

Time is the only judge worth any respect;
'xcept, I don't have much time to wait for it.

— Yours Truly


Thank You!
http://physics1.howard.edu/~thubsch/