Off-Shell Supermultiplets
Error Correcting Codes
& some other surprises

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Error-Corrected Off-Shell Supermultiplets

Program

- Off-Shell Supermultiplets & Classification
  - Off-Shell Supermultiplets: Worldline Perspective
  - Not Your Father's Lie Algebra Representations
  - Supersymmetry is highly degenerate
- Pictures > 10,000 Equations
  - The Adjoint/Fundamental Representation
  - Some Simple Examples: Adinkras
  - Chromotopology and Chromotopography
- Supersymmetry, Error-Correction & More
  - Projections and Their Binary Encryption
  - Constrained and Quotiented Supermultiplets
  - …and ∞ Many Other Supermultiplets

These are not "well-known"

Off-Shell Supermultiplets & Classification

Off-Shell Supermultiplets

- Nature is quantum; we need partition functionals

$$Z[\vartheta] := \int \mathbf{D}[\phi] \; e^{-\frac{i}{\hbar}\left(S[\phi] + \int \mathrm{d}^4\mathrm{x}\, \vartheta\cdot\phi\right)}$$

  - …where the fields must not be subject to any spacetime *differential* equation that could be derived as an equation of motion
  - Non-differential constraints are OK: they do not propagate
- Off-Shell Supermultiplets
  - Off-shell component fields…
  - …that form a complete orbit of the supersymmetry algebra
  - ϕ, $Q_1(\phi)$, $Q_2(\phi)$, … , $Q_1(Q_2(\phi))$, $Q_1(Q_3(\phi))$, … $Q_1(Q_2(\dots Q_N(\phi)))$,
  - …for every ϕ: $(2^{N-1}, 2^{N-1})$ component fields
  - E.g.: 4d spacetime ➔ $N = 4$, (8,8) component fields
    - *…but, chiral superfields are only half as large…*

Off-Shell Supermultiplets & Classification

The Worldline Perspective

- Restrict (dimensionally reduce) to the worldline
  - Must be included in any ($d > 1$ spacetime) physical theory
  - Is present in the Hilbert space of any field theory
  - May well be the underlying M-theory…
    - …extends to the worldsheet [“bow-ties” obstruction; SJG.Jr. & TH]
- Worldline supersymmetry w/o central charges
  - $\{ Q_I , Q_J \} = 2\, \delta_{IJ}\, H$ and $[H , Q_I] = 0$, for all $I, J = 1, 2, 3 \ldots N$.
  - Lorentz symmetry: $Spin(1, d-1)$ ➔ $Spin(1,0) = \mathbf{Z}_2$ (boson/fermion)
    - No rotations, boosts, component field mixing
  - Full $Spin(1, d-1)$ *etc*. may be reconstructed afterwards
    - …by mixing component fields, dimension-by-dimension…
  - $(Q_I)^2 = i\, d/dt$, for each I = 1, 2, 3, …, N.
    - In supermultiplets, $(Q_I)^2 \approx 1$ (fields as Taylor series/towers)

Off-Shell Supermultiplets & Classification

Not Your Father's Lie Algebra Representations

- But… but… *irreps of Lie algebras are well known!*
  - *As well as super-algebras and their on-shell representations!*
  - But, NOT off-shell representations.
- Consider $V_j := \{\ |j,m\rangle,\ |m| \leq j\ \}$ — eigenspace of $\boldsymbol{J}^2$ in $su(2)$,
  - …built from eigenspaces of $\boldsymbol{J}_3$, mutually commuting generators.
- So, in $\{\ Q_I\ ,\ Q_J\ \} = 2\delta_{IJ} H$, $[\ Q_I, H\ ] = 0$,
  - The generator that commutes with everyone is H
  - …but, eigenstates of H satisfy an ODE, ≈ EoM, are *classical*.
  - We could fiber them over the energy-momentum space
  - …(quantum/free fields ≈ sheaves of classical fields)
  - …representations ≈ filtered Clifford supermodules
- Happily, there is a more user-friendly approach

Off-Shell Supermultiplets & Classification

Not Your Father's Lie Algebra Representations

- For the Lie algebra congnoscenti:
  - Standard: Lie algebra = $\{H_\alpha, E_\alpha, E_{\alpha+\beta}, \ldots\}$
  - $[\, H_\alpha \,, E_\alpha \,] = \alpha \cdot E_\alpha$ and $[\, E_\alpha \,, E_\beta \,] = N_{\alpha+\beta} \cdot E_{\alpha+\beta}$ for $\alpha \neq \beta$
  - $[\, H \,, Q_I \,] = \mathbf{0} \cdot E_\alpha$ and $\{\, Q_I \,, Q_J \,\} = \mathbf{0} \cdot Q_{\text{“}I+J\text{”}}$ for $I \neq J$
  - $[\, E_\alpha \,, E_{-\alpha} \,] = 2H_\alpha$ for each α, *vs.* $\{\, Q_I \,, Q_I \,\} = 2H$, for all $I = 1,2,3,\ldots$
- Supersymmetry is a *highly degenerate* graded algebra

Ouch!

- Just in case you are not yet convinced:
  - Write $[\, X_a \,, X_b \,] = i\, f_{ab}{}^c\, X_c$.
  - The Killing metric $g_{ab} := -f_{ac}{}^d\, f_{bd}{}^c$
  - Write $H = X_0$, and $Q_I = X_I$; $f_{IJ}{}^0 = -2i \neq 0$, — all other $f_{ab}{}^c = 0$,
  - …so the Killing metric is *identically zero*.
  - Even $\mathrm{Tr}[\, X_a\, X_b \,] = (\ldots)H$, ➔ODE, ≈ EoM

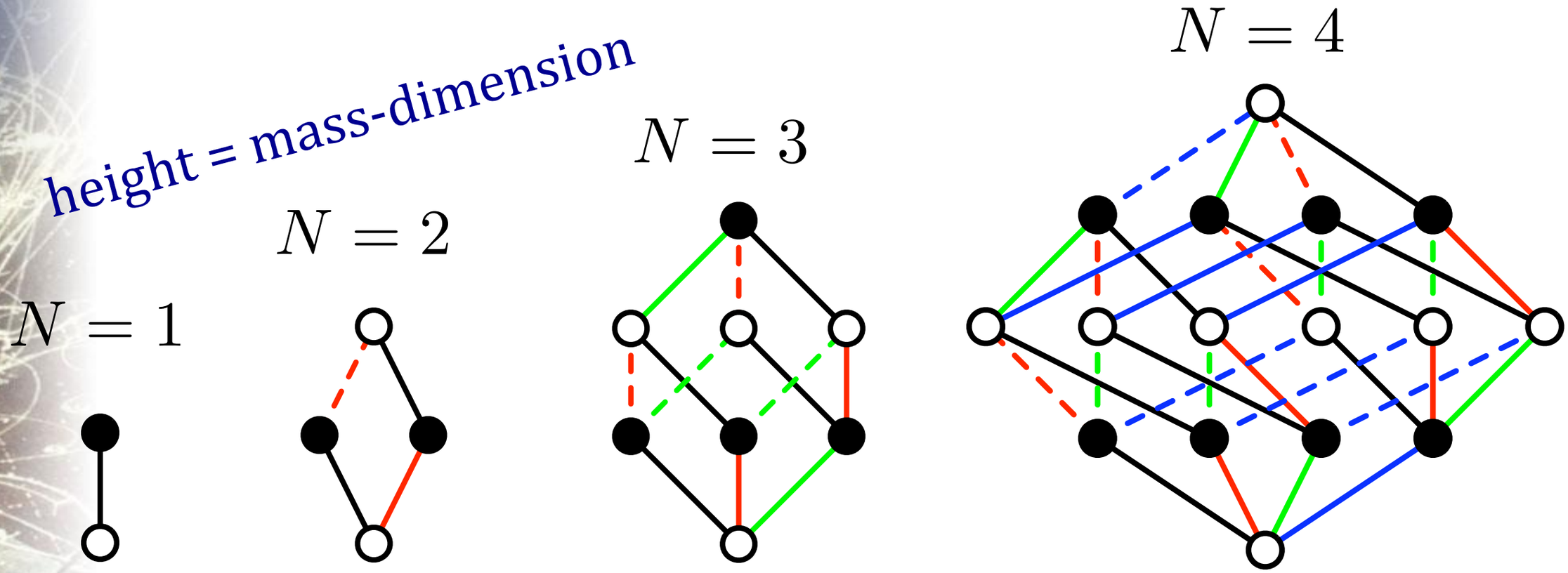
No can do!

Pictures > 10,000 Equations

The Adjoint/Fundamental Supermultiplets

- Since, $\{ Q_I , Q_J \} = 2\delta_{IJ} H$, avoid $(Q_I)^2$ and use
  - ϕ, $Q_1(\phi)$, $Q_2(\phi)$, … , $Q_1(Q_2(\phi))$, $Q_1(Q_3(\phi))$, … $Q_1(Q_2(\dots Q_N(\phi)))$
  - All of the form $\mathbf{Q^b}(\phi)$, where **b** is an N-digit binary number
  - All have the structure (*chromotopography*) of an N-cube

height = mass-dimension

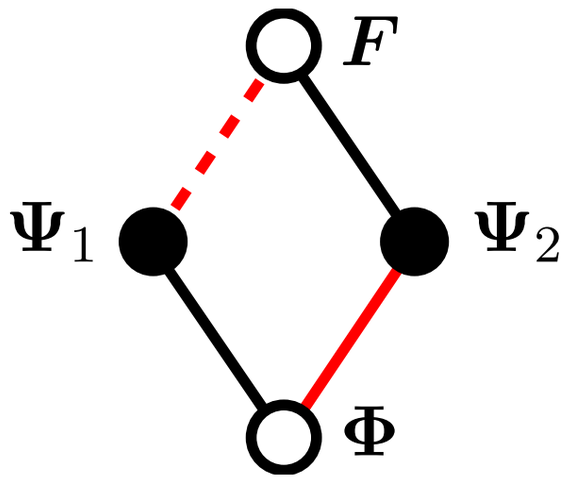


Pictures > 10,000 Equations

In case you were not convinced

- Even just for $N = 2$:

$$Q_1\,\Phi = i\,\Psi_1,$$
$$Q_1\,\Psi_1 = \dot{\Phi},$$
$$Q_1\,\Psi_2 = F,$$
$$Q_1\,F = i\dot{\Psi}_2,$$



$$Q_2\,\Phi = i\,\Psi_2,$$
$$Q_2\,\Psi_1 = -F,$$
$$Q_2\,\Psi_2 = \dot{\Phi},$$
$$Q_2\,F = -i\,\dot{\Psi}_1,$$

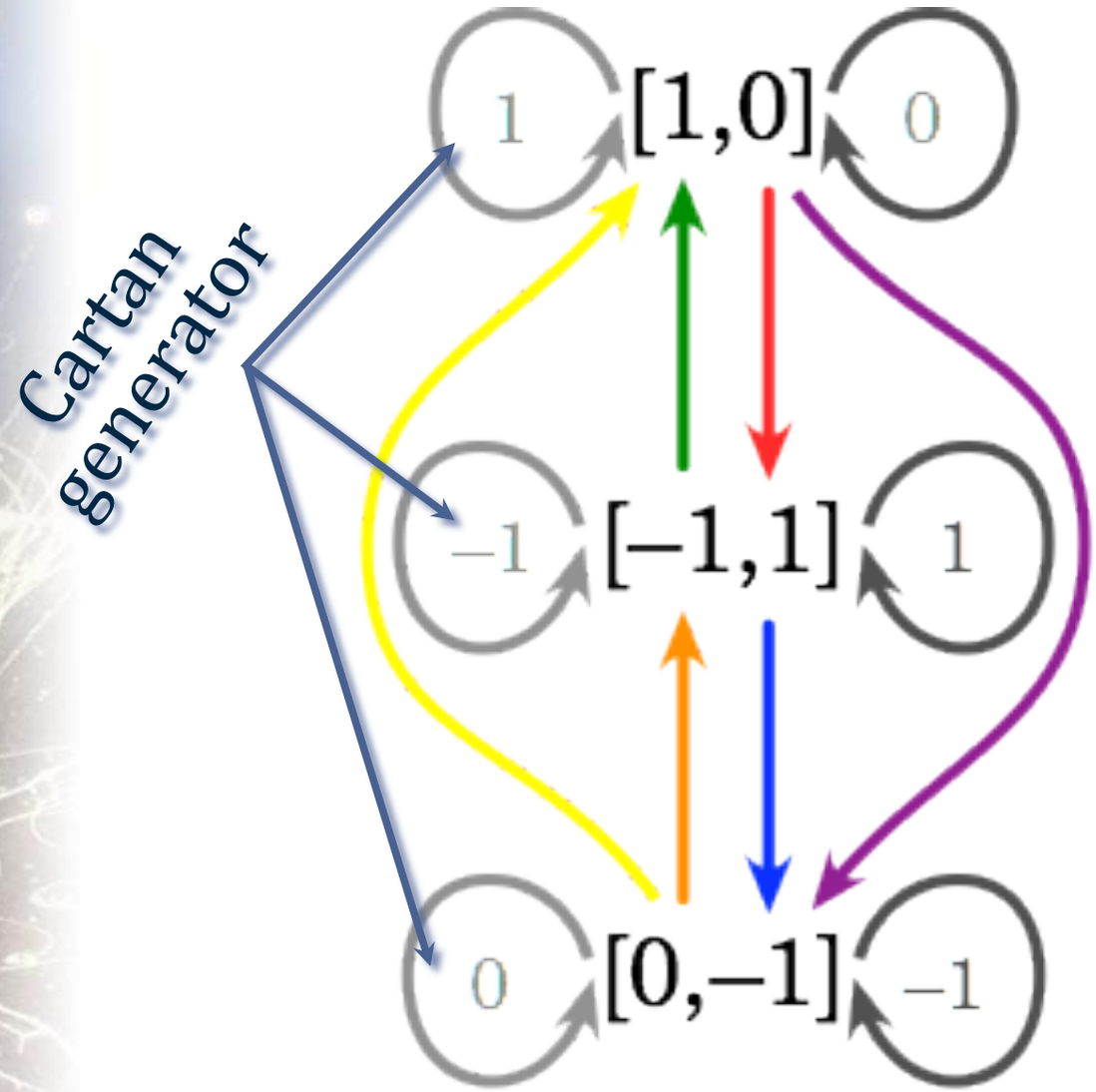
- To be precise, $N{\cdot}2^N$ equations
- …which is >10,000 when $N > 10$.
- …and is 2,048 for $N = 8$ (double supersymmetry in 4d)

- **No need to write them all out.** There will be no quiz.

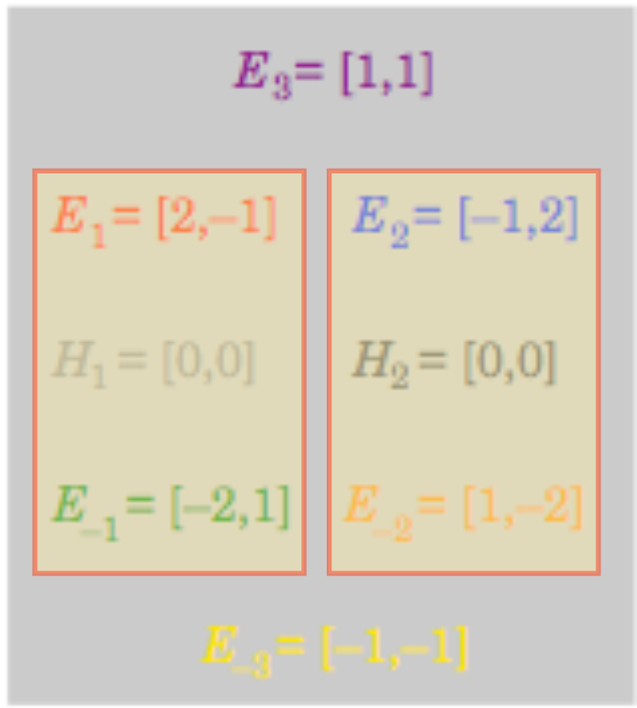
Pictures > 10,000 Equations

Not Your Father's Lie Algebra Representations

- By comparison, the familiar **3** of $su(3)$ would have 3 nodes:



The fundamental, 3-dimensional representation of $su(3)$



raising

Cartan

lowering

The adjoint, 8-dimensional representation of $su(3)$

In Supersymmetry

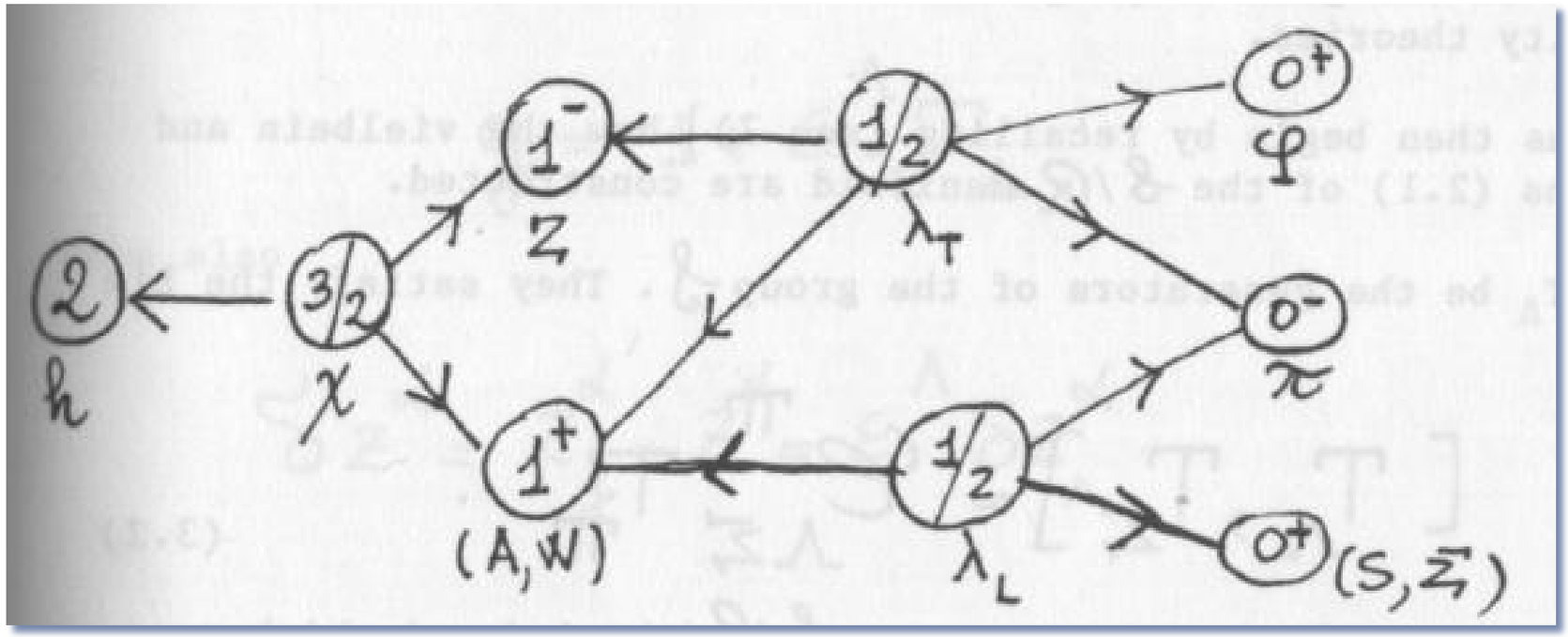
$Q_I^\dagger = Q_I$ raising = lowering

$(Q_I)^2 = H$ (same) degeneracy

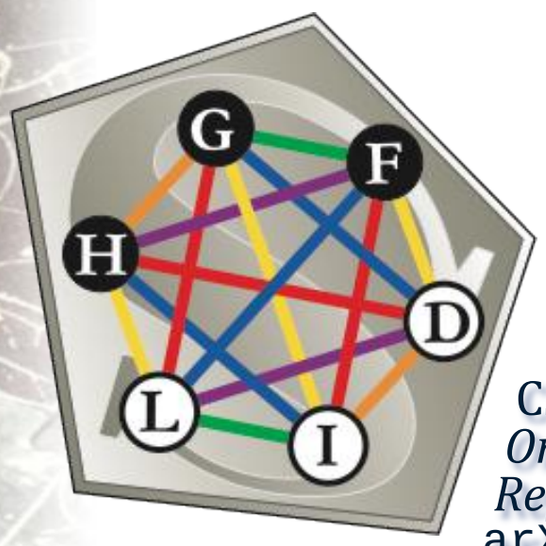
Pictures > 10,000 Equations

Oh, by the way…

- Graphical depiction of supersymmetry transformations
  - (just as many other methods in physics and mathematics)
- …has been done routinely, and all the time
- However, the practice has not been formalized until



Pietro Fré: *Introduction to harmonic expansions on coset manifolds and in particular on coset manifolds with Killing spinors*, in Supersymmetry and supergravity 1984 (Trieste, 1984), p. 324–367, (World Sci. Pub., Singapore, 1984).

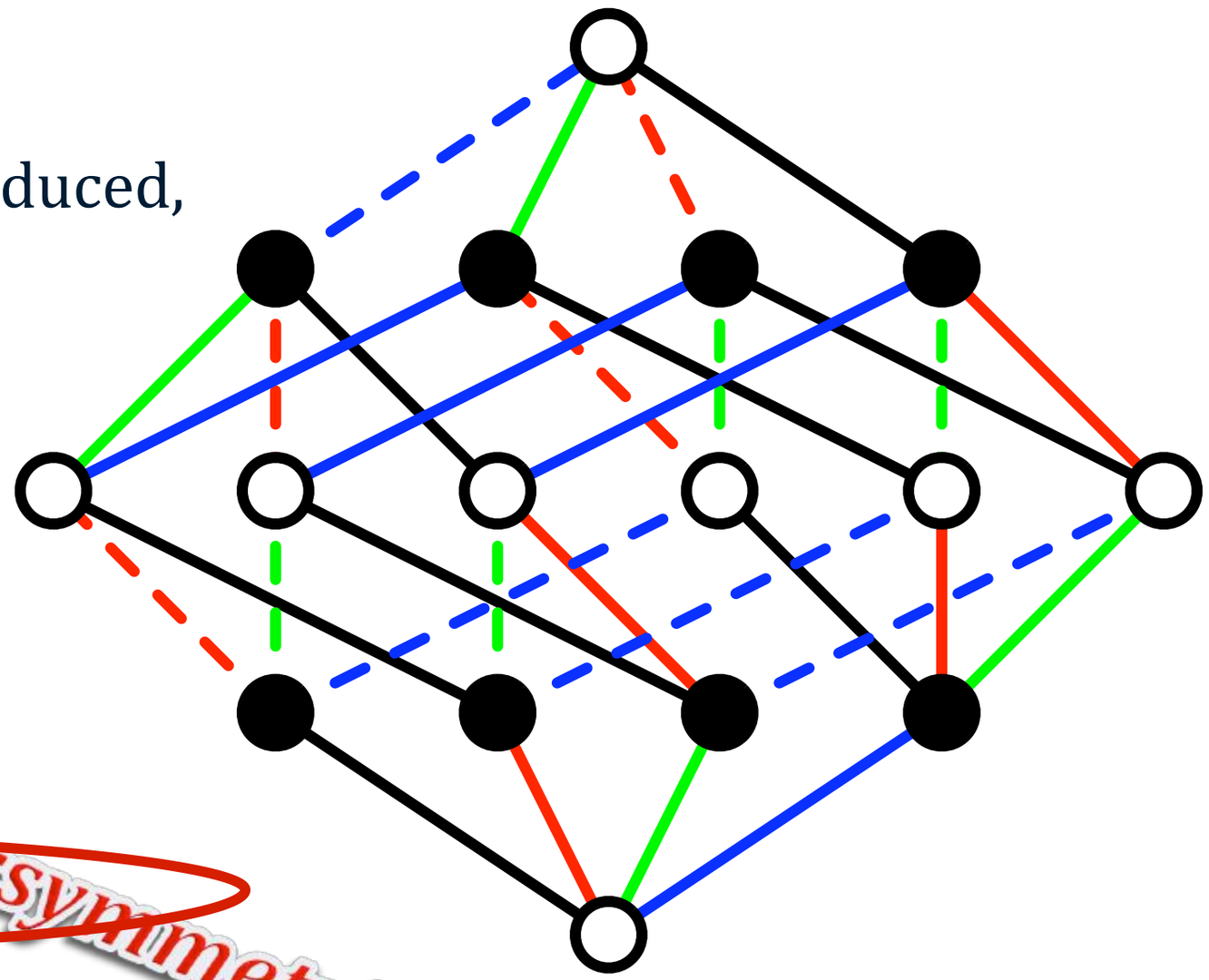


C.F. Doran, M.G. Faux, S.J. Gates, Jr., T. Hübsch, K.M. Iga and G.D. Landweber: *On Graph-Theoretic Identifications of Adinkras, Supersymmetry Representations and Superfields*, *Int. J. Mod. Phys.* **A22** (2007) 869–930, `arXiv:math-ph/0512016`.

Pictures > 10,000 Equations

Some Simple Examples: Adinkras

- 4d spacetime simple supersymmetry: $N = 4$
  - The "vector" superfield
  - B4 Wess-Zumino gauge
  - Real, unconstrained, unreduced, ungauged, unrestricted, un……ed = "intact" supermultiplet
- What can we do with it?
  - Reduce using the D_I's
  - $\{ D_I , D_J \} = 2 \, \delta_{IJ} \, H$, $[H , D_I] = 0$,
  - $\{ Q_I , D_J \} = 0$.
- Write D_I-equations
  - …which are not d/dt-equations.

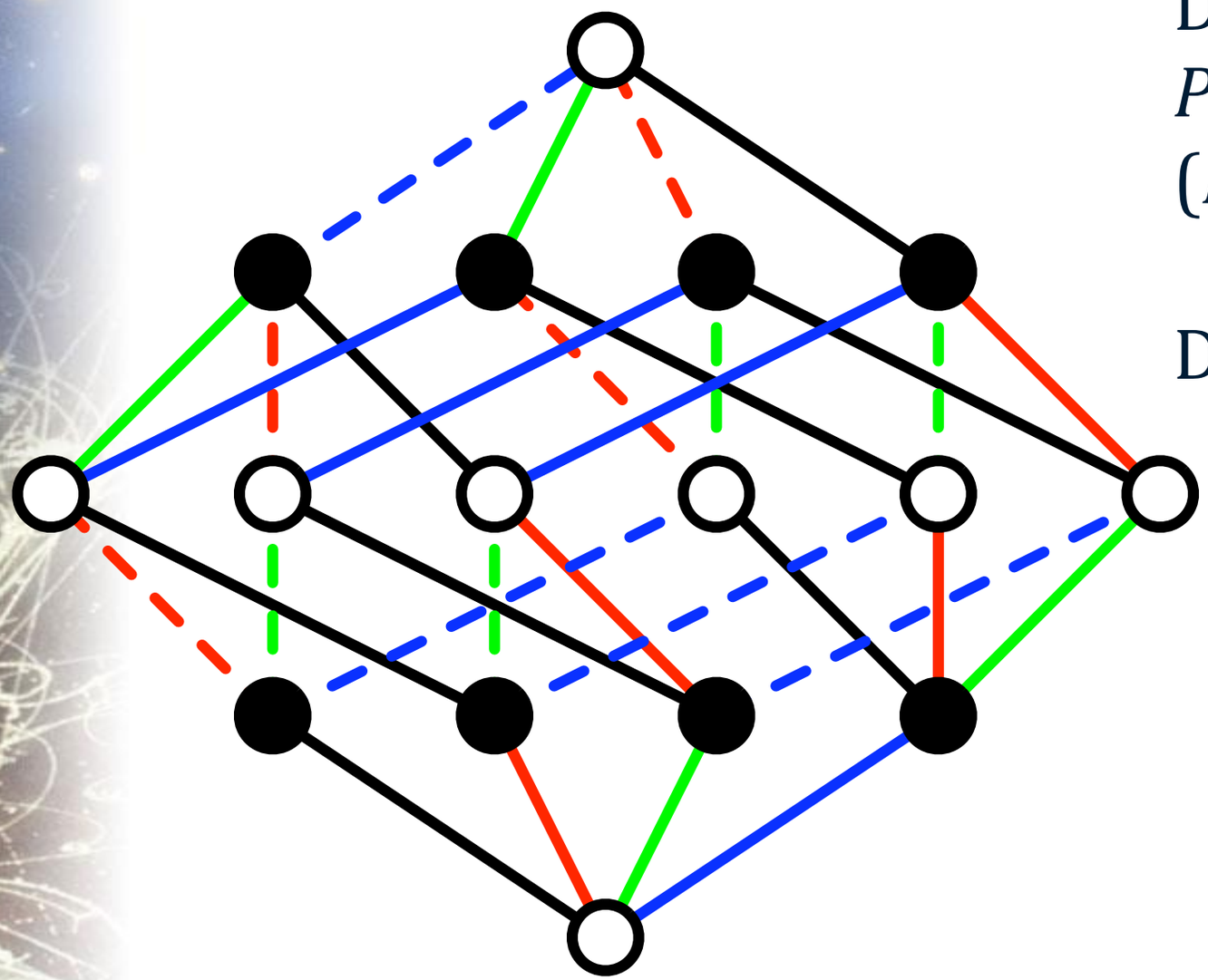


Supersymmetric!

Pictures > 10,000 Equations

Chromotopology and Chromotopography

- For example,

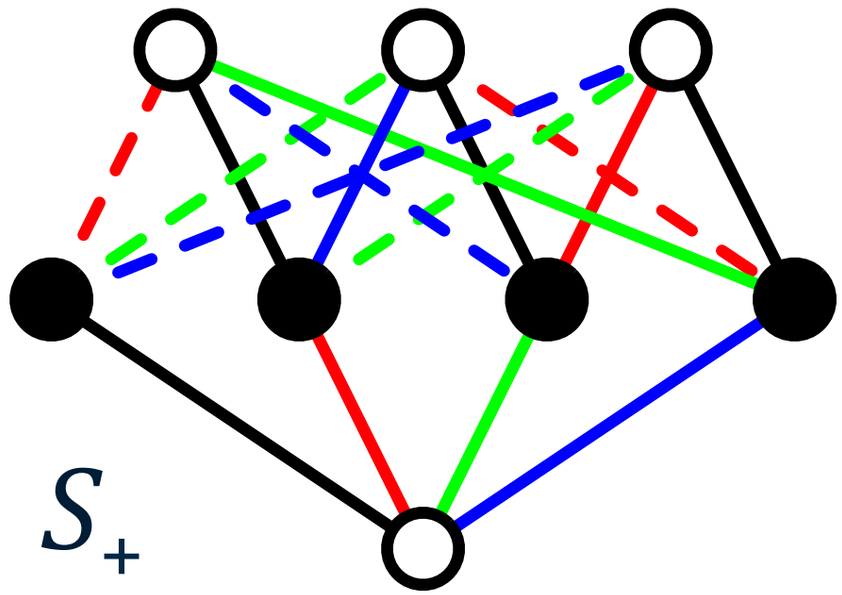


Define *quasi-projectors*

$P_{IJKL}{}^{\pm} := D_I\, D_J \pm \tfrac{1}{2}\, \varepsilon_{IJ}{}^{KL}\, D_K\, D_L$

$(P_{IJKL}{}^{\pm})^2 = H^2 \cdot P_{IJKL}{}^{\pm}$

Define $S_{\pm} := \{\, P_{IJKL}{}^{\pm}(\mathrm{SM}) = 0 \,\}$

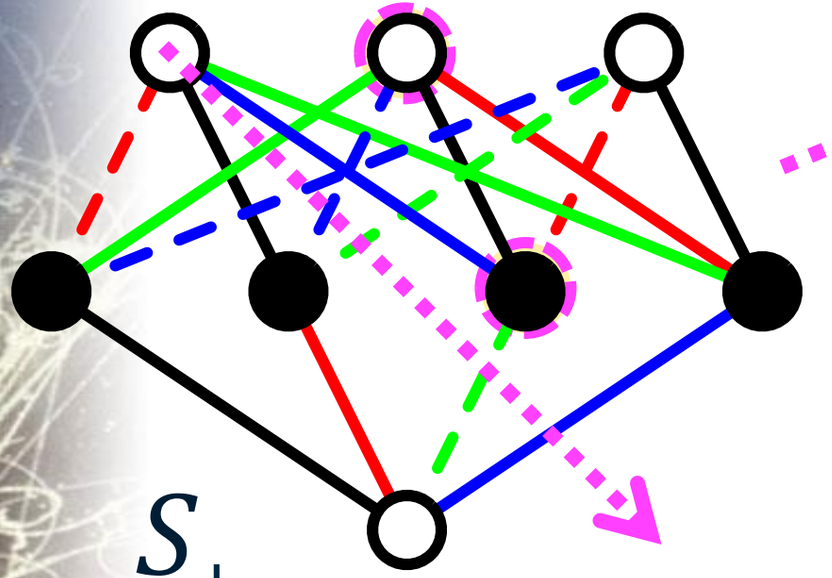


S_+

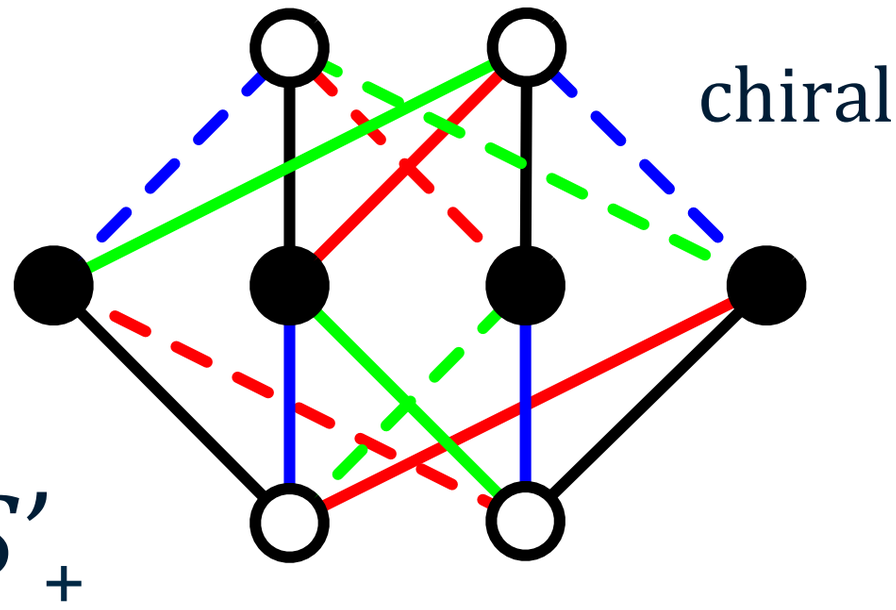
Pictures > 10,000 Equations

Chromotopology and Chromotopography

- Example, cont'd:



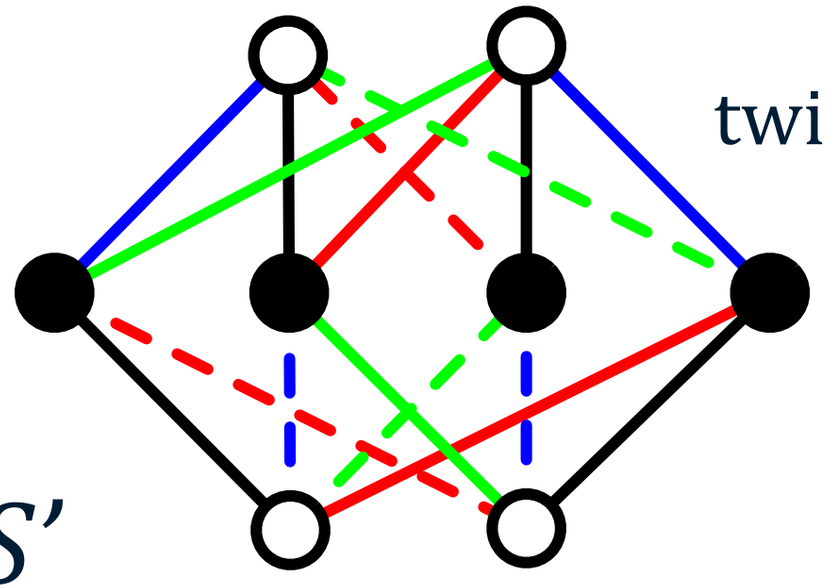
S_+



chiral

S'_+

$$F_1 \to \phi_2 := \int \mathrm{d}t\, F$$



twisted chiral

S'_-

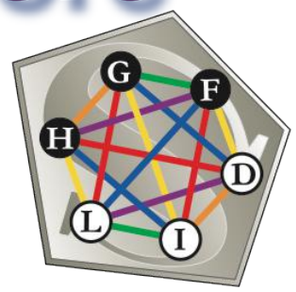
Supersymmetry, Error-Correction & More

Projections and Their Binary Encryption

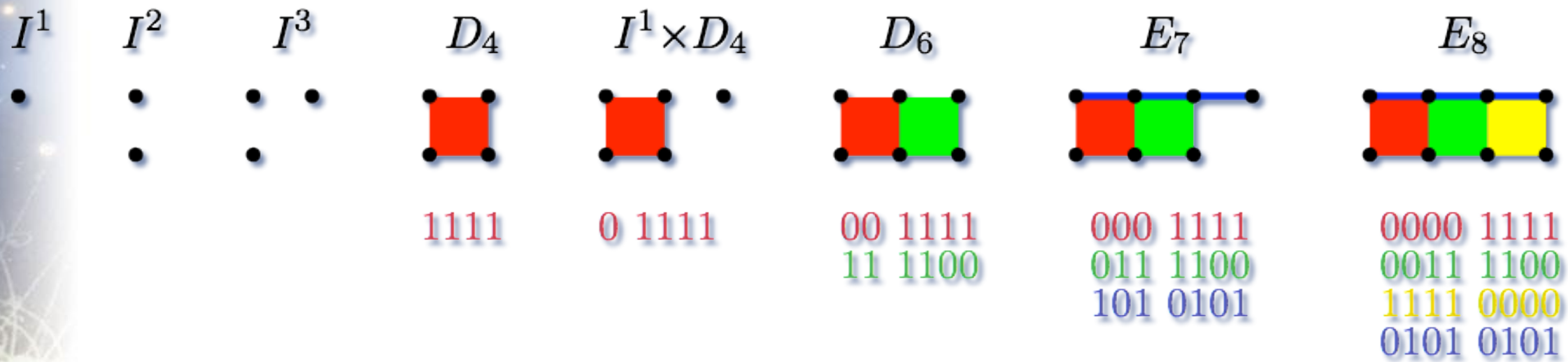
- What about more supersymmetries?
- Impose $P_{IJKL}^{\pm} := D_I D_J \pm \frac{1}{2} \varepsilon_{IJ}^{KL} D_K D_L$ for some fixed I, J, K, L
- …these are quasi-projectors only if $|\{IJ\ldots\}| = 0 \pmod 4$
- …and $[P_{IJKL}^{\pm} , P_{MNPQ}^{\pm}] \approx 0$, if $|\{IJKL\}\cap\{MNPQ\}| = 0 \pmod 2$
- Write $\mathbf{D^b} := D_I D_J D_K D_L$; **b** has 1's at I, J, K, L positions, "0" otherwise
- E.g.: $[111100] = D_1 D_2 D_3 D_4 \approx P_{1234}^{\pm}$, $[110011] \approx P_{1256}^{\pm}$,
- … $[P_{1234}^{\pm} , P_{1256}^{\pm}] = H^2 \cdot P_{3456}^{\pm} \approx 0$.
- That is, [111100]+[110011] = [221111] ≈ [001111]
- Doubly even linear block code d_6: error-correcting encryption (?!)
- Classify these codes = classify "minimal" supermultiplets
- …up to ϕ ➔ $(d\phi/dt)$ and inverse component field redefinitions
- How hard could that be?

Supersymmetry, Error-Correction & More

Projections and Their Binary Encryption

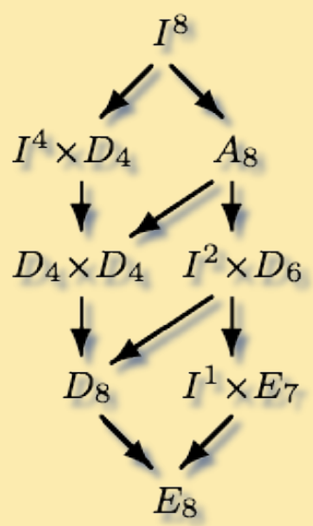


- Let's start:



N							
1	2	3	4	5	6	7	8
I^1 (1;1)	I^2 (2;2)	I^3 (4;4)	I^4 (8;8)	I^5 (16;16)	I^6 (32;32)	I^7 (64;64)	I^8 (128;128)
			D_4 (4;4)	$I^1 \times D_4$ (8;8)	$I^2 \times D_4$ (16;16)	$I^3 \times D_4$ (32;32)	$I^4 \times D_4$ (64;64) A_8 (64;64)
					D_6 (8;8)	$I^1 \times D_6$ (16;16)	$D_4 \times D_4$ (32;32) $I^2 \times D_6$ (32;32)
						E_7 (8;8)	D_8 (16;16) $I^1 \times E_7$ (16;16)
							E_8 (8;8)

The minimal supermultiplets

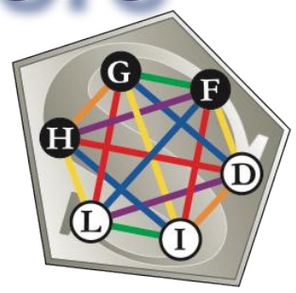


Note the non-uniqueness (& non-linearity) of the sub-code embeddings.

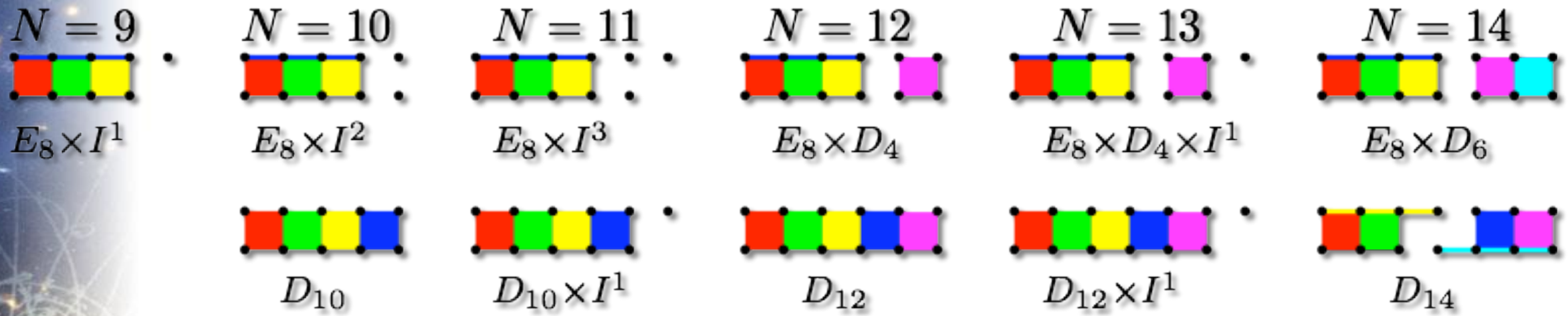
Algebraically distinct ways to construct the same supermultiplet

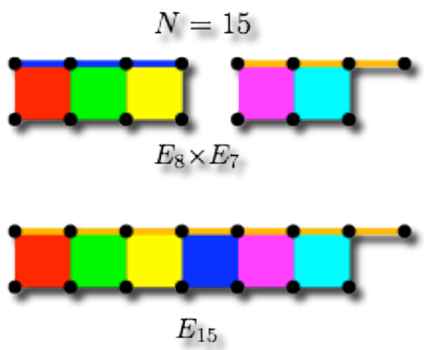
Supersymmetry, Error-Correction & More

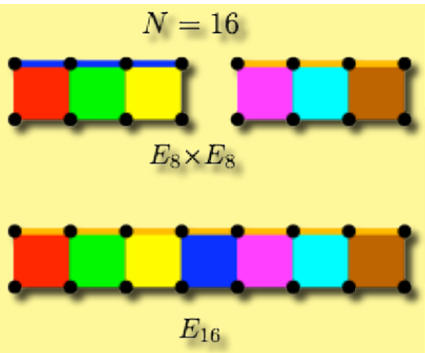
Projections and Their Binary Encryption



- For $N \geq 8$, this becomes even richer:





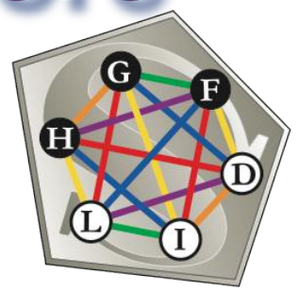


The (well-known) only two even, unimodular lattices

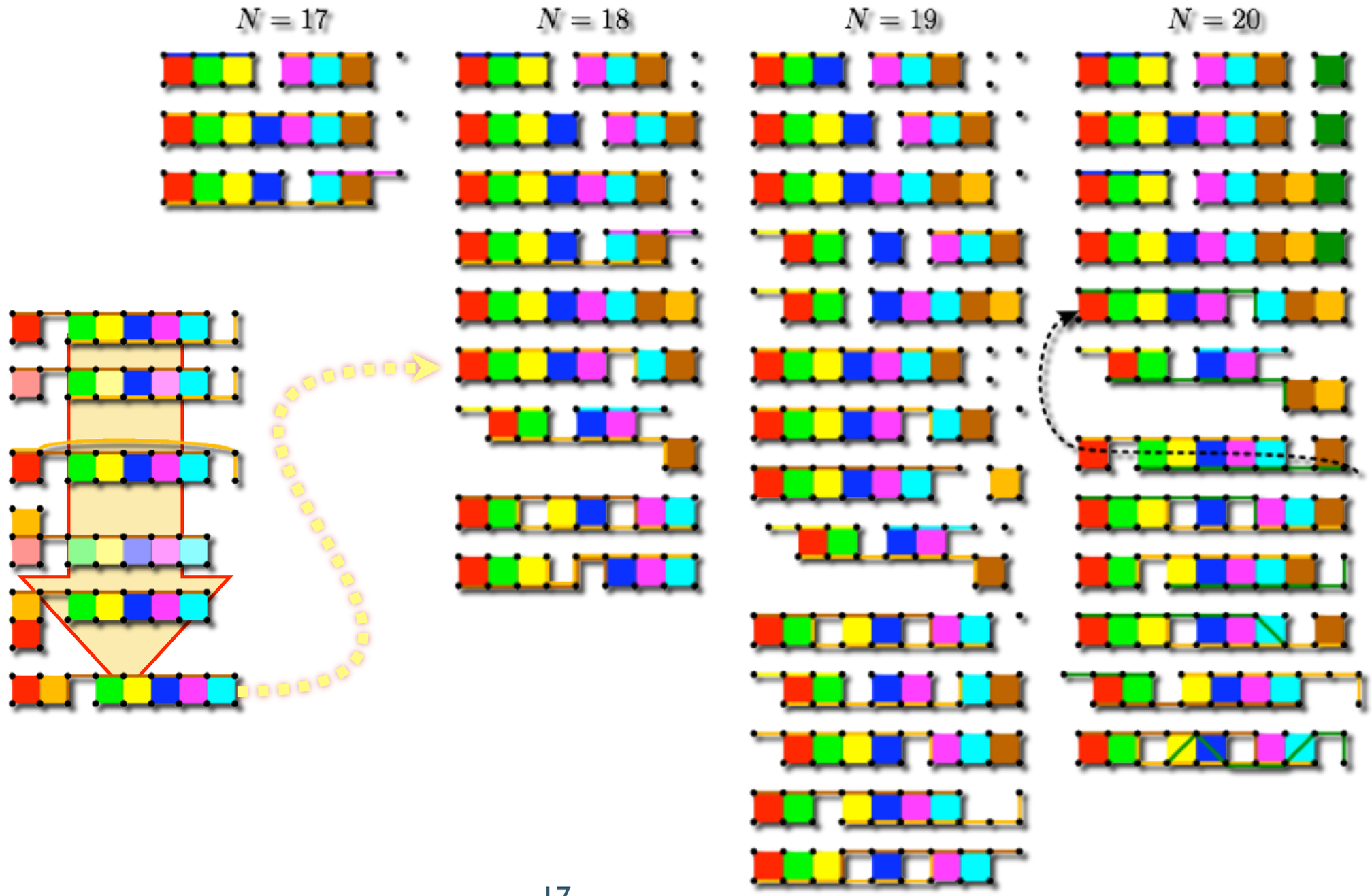
Lie algebras: SO(32) & $E_8 \times E_8$

Supersymmetry, Error-Correction & More

Projections and Their Binary Encryption

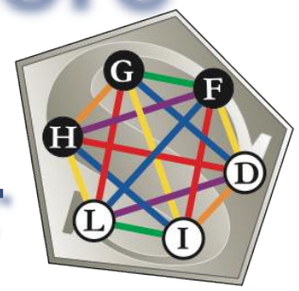


- And then it becomes really, Really, REALLY hard:



Supersymmetry, Error-Correction & More

Projections and Their Binary Encryption



w/Robert Miller

- So, can we classify these codes?

k \ N =	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
1	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#
2	·	·	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#
3	·	·	·	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#
4	·	·	·	·	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	+
5	·	·	·	·	·	·	·	·	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	+	+
6	·	·	·	·	·	·	·	·	·	·	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	#	+	+
7	·	·	·	·	·	·	·	·	·	·	·	#	#	#	#	#	#	#	#	#	#	#	#	#	#	+	+	+	+
8	·	·	·	·	·	·	·	·	·	·	·	·	#	#	#	#	#	#	#	#	#	#	#	#	#	·	·	·	+
9	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	#	#	#	#	#	#	·	·	·	·	·	·	·
10	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	#	#	#	#	#	#	#	+	·	+	·
11	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	#	#	#	#	#	#	+	+	+	+
12	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	#	#	#	#	#	#	+	+	+
13	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	#	#	#	+	+
14	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	#	#	+
15	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	#	#
16	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	#

E_8

$E_8 \times E_8$, E_{16}

10^{12} codes

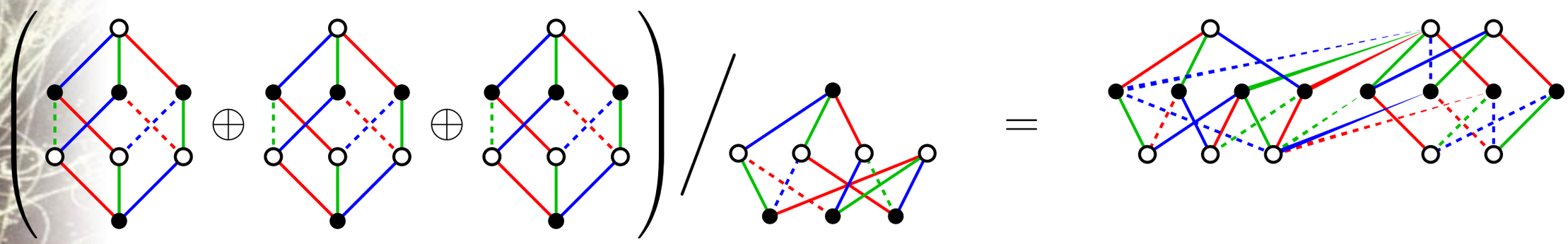
9 codes

85 codes

Supersymmetry, Error-Correction & More

Constrained and Quotiented Supermultiplets

- More generally,
  - With two off-shell supermultiplets, $\mathbf{D^b}$: **X** ➔ **Y**, define:
  - Kernel, **A** := { **X**: $\mathbf{D^b}$(**X**) = 0 }, (e.g., chiral supermultiplet)
  - Cokernel, **B** := { **Y** (mod $\mathbf{D^b}$(**X**)) }, (e.g., vector sm. in WZ gauge)
- Already for N = 3,
  - This defines new (and ever bigger) supermultiplets:

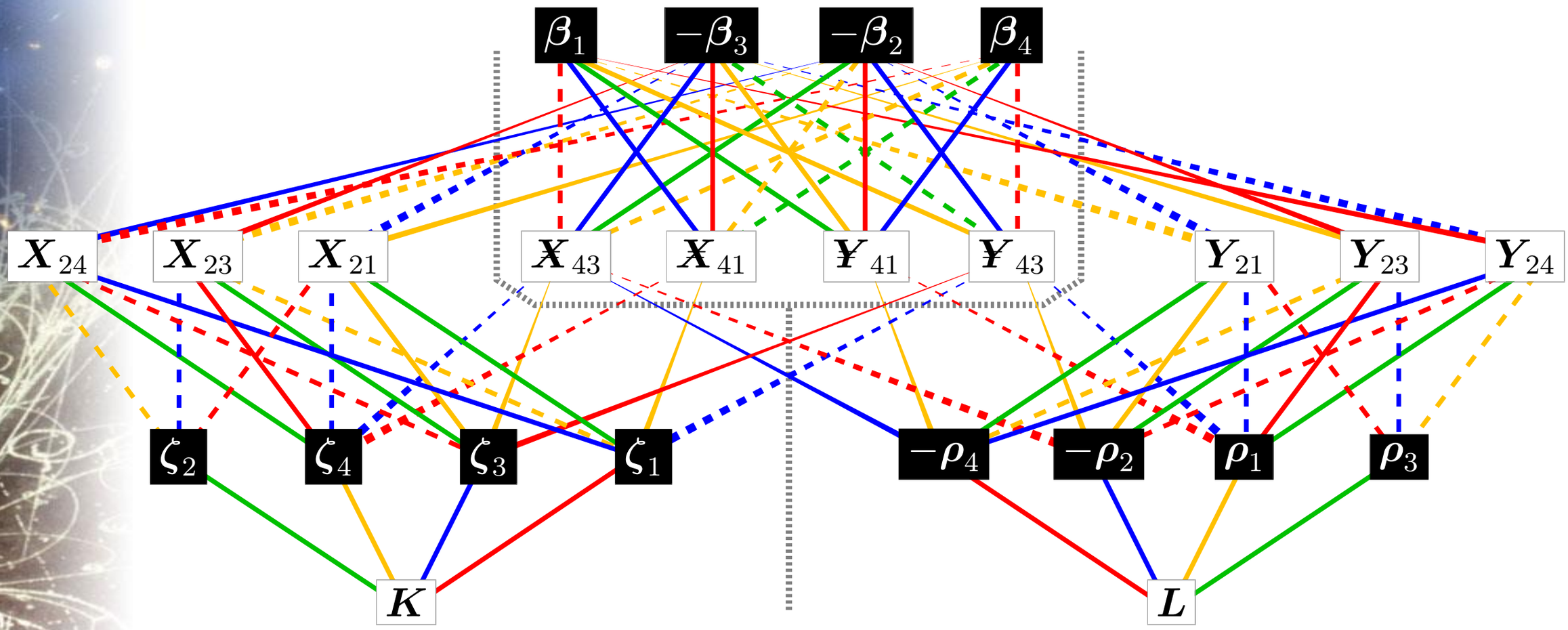


  - ...depicts (3 **Y**)/(D_I **X**) = a new (5|8|3)-component supermultiplet
  - Would you have preferred the $3\cdot(3\cdot2^3) = 72$ equations?

Supersymmetry, Error-Correction & More

Constrained and Quotiented Supermultiplets

- And, an $N = 4$ (in fact, 4d simple) supersymmetry example:
  - The "complex linear supermultiplet/superfield" [SJG.Jr, JH, TH & KS]:

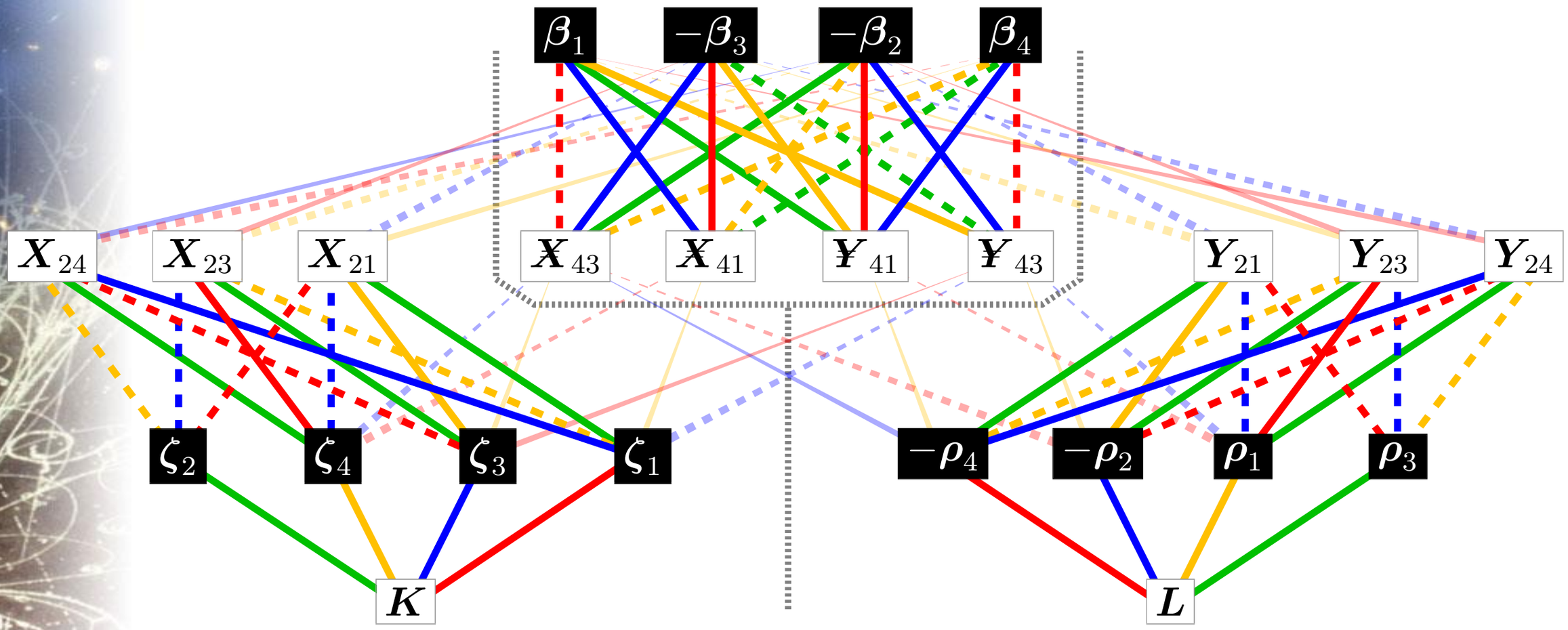


- Bigger supermultiplets = networks of Adinkras, connected by one-way (BRST-like) supersymmetry transformations

Supersymmetry, Error-Correction & More

Constrained and Quotiented Supermultiplets

- And, an $N = 4$ (in fact, 4d simple) supersymmetry example:
  - The "complex linear supermultiplet/superfield" [SJG.Jr, JH, TH & KS]:

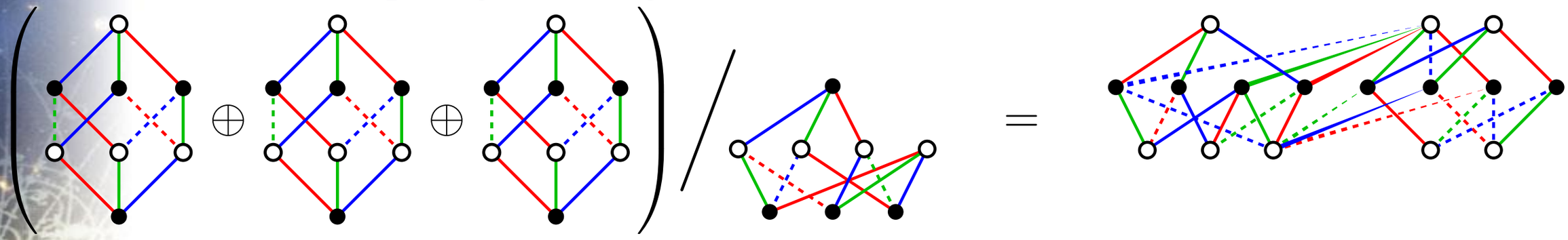


- Bigger supermultiplets = networks of Adinkras, connected by one-way (BRST-like) supersymmetry transformations

Supersymmetry, Error-Correction & More

...and ∞ Many Other Supermultiplets

- Weyl Construction
  - Lie algebras: $\boldsymbol{R}_1 \otimes \boldsymbol{R}_2 = \boldsymbol{R}_3 \oplus \boldsymbol{R}_4 \oplus \ldots \oplus \boldsymbol{R}_k$ and $\boldsymbol{R}_1 \oplus \boldsymbol{R}_2$ unique.
  - Not so in supersymmetry:



  - Thus, $\boldsymbol{Y}_1 \oplus \boldsymbol{Y}_2 \oplus \boldsymbol{Y}_3$ reduces also *differently*; it is not unique.
- In turn, already for $N = 1$,

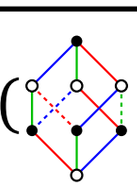
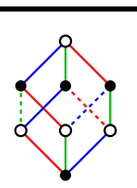
$$\begin{bmatrix}\psi'\\ \phi'\end{bmatrix} \otimes \begin{bmatrix}\psi''\\ \phi''\end{bmatrix} \Rightarrow \begin{bmatrix}(\psi'\phi''+\phi'\psi'')\\ (\phi'\phi'')\end{bmatrix} \oplus \left\{ \begin{bmatrix}(2i\psi'\psi''-\phi'\overset{\leftrightarrow}{\partial}_\tau\phi'')\\ (\psi'\phi''-\phi'\psi'')\end{bmatrix} + \alpha \begin{bmatrix}\partial_\tau(\phi'\phi'')\\ (\psi'\phi''+\phi'\psi'')\end{bmatrix} \right\}$$

The parametrization is this simple only in the simplest, $N = 1$ case

Supersymmetry, Error-Correction & More

...and ∞ Many Other Supermultiplets

- Weyl-esque Construction:
  - Form $\boldsymbol{Y}_1 \oplus \boldsymbol{Y}_2$, and reduce as $\mathbf{D^b}$-constraining/gauging [GK & TH]:

	Lie algebra irrreps (Weyl construction)	**Off-shell supermultiplets**
Starting object(s) and their depiction	fundamental irrep ($\square$)	Adinkras ( &  & $\cdots$)
Combining operator	$\otimes$	$\oplus$
Reduction methods	Young symmetrization, traces w/inv. tensors	Construction 3.1: $\ker(\mu)$ and $\mathrm{cok}(\mu)$ of supersymmetric maps
Resulting objects and their depiction	arbitrarily large irreps, & their Young tableaux (1-quadrant graphs)	networks of otherwise proper Adinkras, connected by one-way Q-action edges

- Indefinite, but unlike Lie algebras, no proof of completeness

- Adinkras, off-shell supermultiplets/superfields & classification:
  - arXiv : math-ph/0512016, hep-th/0611060, 0806.0051, 0806.0050, 0811.3410, 0901.4970, 1108.4124.
- From worldline to worldsheets and beyond:
  - arXiv : 1104.0722, 1104.3135, 1208.5999.
- Constructing actions using Adinkras
  - arXiv : hep-th/0605269, 0803.3434,
- Symmetries and structures in Adinkras:
  - arXiv : 0904.4719, 1202.4418, 1203.4189, 1210.0478 .
- Reducibility and new supermultiplets:
  - arXiv : 0809.5279, 0904.4719, 1202.4342.
- Rigorous mathematical foundation:
  - arXiv : math-ph/0603012.

Thanks!

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